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Again on the Order on a Special Polygon[1]

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The articles [11], [15], [3], [2], [14], [4], [8], [12], [1], [10], [6], [7], [9], [5], and [13] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we follow the rules: D is a non empty set, f is a finite sequence of elements of D, g is a circular finite sequence of elements of D, and p, p_1, p_2, p_3, q are elements of D.

The following propositions are true:

(1) If $q \in \operatorname{rng}(f \upharpoonright p \hookleftarrow f)$, then $q \hookleftarrow f \leq p \hookleftarrow f$.

(2) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \hookleftarrow f \leq q \hookleftarrow f$, then $q \hookleftarrow (f :- p) = (q \hookleftarrow f - p \hookleftarrow f) + 1$.

(3) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \hookleftarrow f \leq q \hookleftarrow f$, then $p \hookleftarrow (f -: q) = p \hookleftarrow f$.

(4) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \hookleftarrow f \leq q \hookleftarrow f$, then $q \hookleftarrow (f \circlearrowleft p) = (q \hookleftarrow f - p \hookleftarrow f) + 1$.

(5) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \hookleftarrow f \leq p_2 \hookleftarrow f$ and $p_2 \hookleftarrow f < p_3 \hookleftarrow f$, then $p_2 \hookleftarrow (f \circlearrowleft p_1) < p_3 \hookleftarrow (f \circlearrowleft p_1)$.

(6) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \hookleftarrow f < p_2 \hookleftarrow f$ and $p_2 \hookleftarrow f \leq p_3 \hookleftarrow f$, then $p_2 \hookleftarrow (f \circlearrowleft p_1) \leq p_3 \hookleftarrow (f \circlearrowleft p_1)$.

(7) If $p \in \operatorname{rng} g$ and $\operatorname{len} g > 1$, then $p \hookleftarrow g < \operatorname{len} g$.

2. Ordering of Special Points on a Standard Special Sequence

We follow the rules: f denotes a non constant standard special circular sequence and p, p_1, p_2, p_3, q denote points of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(8) $f_{\lfloor 1}$ is one-to-one.

(9) If $1 < q \hookleftarrow f$ and $q \in \operatorname{rng} f$, then $f_1 \hookleftarrow (f \circlearrowleft q) = (\operatorname{len} f + 1) - q \hookleftarrow f$.

(10) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \hookleftarrow f < q \hookleftarrow f$, then $p \hookleftarrow (f \circlearrowleft q) = (\operatorname{len} f + p \hookleftarrow f) - q \hookleftarrow f$.

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(11) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \looparrowleft f < p_2 \looparrowleft f$ and $p_2 \looparrowleft f < p_3 \looparrowleft f$, then $p_3 \looparrowleft (f \circlearrowleft p_2) < p_1 \looparrowleft (f \circlearrowleft p_2)$.

(12) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \looparrowleft f < p_2 \looparrowleft f$ and $p_2 \looparrowleft f < p_3 \looparrowleft f$, then $p_1 \looparrowleft (f \circlearrowleft p_3) < p_2 \looparrowleft (f \circlearrowleft p_3)$.

(13) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \looparrowleft f \leq p_2 \looparrowleft f$ and $p_2 \looparrowleft f < p_3 \looparrowleft f$, then $p_1 \looparrowleft (f \circlearrowleft p_3) \leq p_2 \looparrowleft (f \circlearrowleft p_3)$.

(14) $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(15) $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(16) $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(17) $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(18) $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(19) $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(20) $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

(21) $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < \operatorname{len} f$.

3. Ordering of Special Points on a Clockwise Oriented Sequence

In the sequel z is a clockwise oriented non constant standard special circular sequence.

We now state a number of propositions:

(22) If $f_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < (\mathrm{W_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f$.

(23) If $f_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f > 1$.

(24) If $z_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(25) If $z_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(26) If $z_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(27) If $z_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(28) If $z_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(29) If $z_1 = \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(30) If $f_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f < (\mathrm{S_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f$.

(31) If $f_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(f))) \looparrowleft f > 1$.

(32) If $z_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(33) If $z_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(34) If $z_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \looparrowleft z$.

(35) If $z_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(36) If $z_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(37) If $z_1 = \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(38) If $f_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f < (\mathrm{E_{min}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f$.

(39) If $f_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f > 1$.

(40) If $z_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(41) If $z_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(42) If $z_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(43) If $z_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(44) If $z_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(45) If $z_1 = \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(46) If $f_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{N_{max}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{E_{max}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f < (\mathrm{E_{max}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f$.

(47) If $z_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(48) If $z_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(49) If $z_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(50) If $z_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(51) If $z_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(52) If $z_1 = \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(53) If $f_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{E_{min}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{S_{max}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f < (\mathrm{S_{max}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f$.

(54) If $z_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(55) If $z_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(56) If $z_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(57) If $z_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(58) If $z_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(59) If $z_1 = \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(60) If $f_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{S_{min}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{W_{min}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f < (\mathrm{W_{min}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f$.

(61) If $z_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(62) If $z_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(63) If $z_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(64) If $z_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(65) If $z_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(66) If $z_1 = \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(67) If $f_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{W_{max}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{N_{min}}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{W_{max}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f < (\mathrm{N_{min}}(\widetilde{\mathcal{L}}(f))) \hookleftarrow f$.

(68) If $z_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(69) If $z_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(70) If $z_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(71) If $z_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(72) If $z_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{max}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

(73) If $z_1 = \mathrm{W_{max}}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z < (\mathrm{W_{min}}(\widetilde{\mathcal{L}}(z))) \hookleftarrow z$.

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