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# Special Polygons

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The articles [13], [2], [9], [1], [4], [3], [14], [10], [12], [6], [7], [8], [11], and [5] provide the notation and terminology for this paper.

## 1. Segments in $\mathcal{E}^2_{\mathrm{T}}$

For simplicity, we follow the rules: $P$ denotes a subset of $\mathcal{E}^2_{\mathrm{T}}$, $f$, $f_1$, $f_2$, $g$ denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, $p$, $p_1$, $p_2$, $q$, $q_1$, $q_2$ denote points of $\mathcal{E}^2_{\mathrm{T}}$, $r_1$, $r_2$, $r'_1$, $r'_2$ denote real numbers, and $i$, $j$, $k$, $n$ denote natural numbers.

The following propositions are true:

(1) For all real numbers $r_1$, $r_2$, $r'_1$, $r'_2$ such that $[r_1, r_2] = [r'_1, r'_2]$ holds $r_1 = r'_1$ and $r_2 = r'_2$.

(2) If $i + j = \mathrm{len}\, f$, then $\mathcal{L}(f, i) = \mathcal{L}(\mathrm{Rev}(f), j)$.

(3) If $i + 1 \leq \mathrm{len}(f \restriction n)$, then $\mathcal{L}(f \restriction n, i) = \mathcal{L}(f, i)$.

(4) If $n \leq \mathrm{len}\, f$ and $1 \leq i$, then $\mathcal{L}(f_{\restriction n}, i) = \mathcal{L}(f, n + i)$.

(5) If $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f - n$, then $\mathcal{L}(f_{\restriction n}, i) = \mathcal{L}(f, n + i)$.

(6) If $i + 1 \leq \mathrm{len}\, f$, then $\mathcal{L}(f \frown g, i) = \mathcal{L}(f, i)$.

(7) If $1 \leq i$, then $\mathcal{L}(f \frown g, \mathrm{len}\, f + i) = \mathcal{L}(g, i)$.

(8) If $f$ is non empty and $g$ is non empty, then $\mathcal{L}(f \frown g, \mathrm{len}\, f) = \mathcal{L}(f_{\mathrm{len}\, f}, g_1)$.

(9) If $i + 1 \leq \mathrm{len}(f -: p)$, then $\mathcal{L}(f -: p, i) = \mathcal{L}(f, i)$.

(10) If $p \in \mathrm{rng}\, f$, then $\mathcal{L}(f :- p, i + 1) = \mathcal{L}(f, i + p \leftarrow\!\!\shortmid f)$.

(11) $\widetilde{\mathcal{L}}(\varepsilon_{(\text{the carrier of } \mathcal{E}^2_{\mathrm{T}})}) = \emptyset$.

(12) $\widetilde{\mathcal{L}}(\langle p \rangle) = \emptyset$.

(13) If $p \in \widetilde{\mathcal{L}}(f)$, then there exists $i$ such that $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$.

(14) If $p \in \widetilde{\mathcal{L}}(f)$, then there exists $i$ such that $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $p \in \mathcal{L}(f_i, f_{i+1})$.

(15) If $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $p \in \mathcal{L}(f_i, f_{i+1})$, then $p \in \widetilde{\mathcal{L}}(f)$.

(16) If $1 \leq i$ and $i+1 \leq \operatorname{len} f$, then $\mathcal{L}(f_i, f_{i+1}) \subseteq \widetilde{\mathcal{L}}(f)$.

(17) If $p \in \mathcal{L}(f,i)$, then $p \in \widetilde{\mathcal{L}}(f)$.

(18) If $\operatorname{len} f \geq 2$, then $\operatorname{rng} f \subseteq \widetilde{\mathcal{L}}(f)$.

(19) If $f$ is non empty, then $\widetilde{\mathcal{L}}(f \frown \langle p \rangle) = \widetilde{\mathcal{L}}(f) \cup \mathcal{L}(f_{\operatorname{len} f}, p)$.

(20) If $f$ is non empty, then $\widetilde{\mathcal{L}}(\langle p \rangle \frown f) = \mathcal{L}(p, f_1) \cup \widetilde{\mathcal{L}}(f)$.

(21) $\widetilde{\mathcal{L}}(\langle p, q \rangle) = \mathcal{L}(p, q)$.

(22) $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(\operatorname{Rev}(f))$.

(23) If $f_1$ is non empty and $f_2$ is non empty, then $\widetilde{\mathcal{L}}(f_1 \frown f_2) = \widetilde{\mathcal{L}}(f_1) \cup \mathcal{L}((f_1)_{\operatorname{len} f_1}, (f_2)_1) \cup \widetilde{\mathcal{L}}(f_2)$.

(25)[1] If $q \in \operatorname{rng} f$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(f -: q) \cup \widetilde{\mathcal{L}}(f :- q)$.

(26) If $p \in \mathcal{L}(f,n)$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(\operatorname{Ins}(f,n,p))$.

## 2. Special Sequences in $\mathcal{E}^2_{\mathrm{T}}$

One can verify the following observations:

* there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is special sequence,
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is special sequence is also one-to-one, unfolded, s.n.c., special, and non trivial,
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one, unfolded, s.n.c., special, and non trivial is also special sequence, and
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is special sequence is also non empty.

Let us mention that there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one, unfolded, s.n.c., special, and non trivial.

We now state the proposition

(27) If $\operatorname{len} f \leq 2$, then $f$ is unfolded.

Let $f$ be an unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider $n$. Observe that $f \restriction n$ is unfolded and $f_{\downharpoonleft n}$ is unfolded.

One can prove the following proposition

(28) If $p \in \operatorname{rng} f$ and $f$ is unfolded, then $f :- p$ is unfolded.

Let $f$ be an unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider $p$. One can check that $f -: p$ is unfolded.

The following propositions are true:

(29) If $f$ is unfolded, then $\operatorname{Rev}(f)$ is unfolded.

(30) If $g$ is unfolded and $\mathcal{L}(p, g_1) \cap \mathcal{L}(g, 1) = \{g_1\}$, then $\langle p \rangle \frown g$ is unfolded.

(31) If $f$ is unfolded and $k+1 = \operatorname{len} f$ and $\mathcal{L}(f,k) \cap \mathcal{L}(f_{\operatorname{len} f}, p) = \{f_{\operatorname{len} f}\}$, then $f \frown \langle p \rangle$ is unfolded.

(32) Suppose $f$ is unfolded and $g$ is unfolded and $k+1 = \operatorname{len} f$ and $\mathcal{L}(f,k) \cap \mathcal{L}(f_{\operatorname{len} f}, g_1) = \{f_{\operatorname{len} f}\}$ and $\mathcal{L}(f_{\operatorname{len} f}, g_1) \cap \mathcal{L}(g, 1) = \{g_1\}$. Then $f \frown g$ is unfolded.

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[1] The proposition (24) has been removed.

(33) If $f$ is unfolded and $p \in \mathcal{L}(f,n)$, then $\mathrm{Ins}(f,n,p)$ is unfolded.

(34) If $\mathrm{len}\, f \leq 2$, then $f$ is s.n.c..

Let $f$ be a s.n.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider $n$. Observe that $f \restriction n$ is s.n.c. and $f_{\lfloor n}$ is s.n.c..

Let $f$ be a s.n.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider $p$. One can verify that $f -: p$ is s.n.c..

One can prove the following propositions:

(35) If $p \in \mathrm{rng}\, f$ and $f$ is s.n.c., then $f :- p$ is s.n.c..

(36) If $f$ is s.n.c., then $\mathrm{Rev}(f)$ is s.n.c..

(37) Suppose that

(i) $f$ is s.n.c.,

(ii) $g$ is s.n.c.,

(iii) $\widetilde{\mathcal{L}}(f)$ misses $\widetilde{\mathcal{L}}(g)$,

(iv) for every $i$ such that $1 \leq i$ and $i+2 \leq \mathrm{len}\, f$ holds $\mathcal{L}(f,i)$ misses $\mathcal{L}(f_{\mathrm{len}\, f}, g_1)$, and

(v) for every $i$ such that $2 \leq i$ and $i+1 \leq \mathrm{len}\, g$ holds $\mathcal{L}(g,i)$ misses $\mathcal{L}(f_{\mathrm{len}\, f}, g_1)$.

Then $f \frown g$ is s.n.c..

(38) If $f$ is unfolded and s.n.c. and $p \in \mathcal{L}(f,n)$ and $p \notin \mathrm{rng}\, f$, then $\mathrm{Ins}(f,n,p)$ is s.n.c..

Let us observe that $\varepsilon_{(\text{the carrier of } \mathcal{E}^2_{\mathrm{T}})}$ is special.

Next we state two propositions:

(39) $\langle p \rangle$ is special.

(40) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\langle p, q \rangle$ is special.

Let $f$ be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider $n$. Observe that $f \restriction n$ is special and $f_{\lfloor n}$ is special.

Next we state the proposition

(41) If $p \in \mathrm{rng}\, f$ and $f$ is special, then $f :- p$ is special.

Let $f$ be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider $p$. One can check that $f -: p$ is special.

Next we state four propositions:

(42) If $f$ is special, then $\mathrm{Rev}(f)$ is special.

(44)[2] If $f$ is special and $p \in \mathcal{L}(f,n)$, then $\mathrm{Ins}(f,n,p)$ is special.

(45) If $q \in \mathrm{rng}\, f$ and $1 \neq q \leftarrow\!\!\!\!\!\!\circ\; f$ and $q \leftarrow\!\!\!\!\!\!\circ\; f \neq \mathrm{len}\, f$ and $f$ is unfolded and s.n.c., then $\widetilde{\mathcal{L}}(f -: q) \cap \widetilde{\mathcal{L}}(f :- q) = \{q\}$.

(46) If $p \neq q$ and if $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\langle p, q \rangle$ is special sequence.

A S-sequence in $\mathbb{R}^2$ is special sequence finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.

We now state several propositions:

(47) For every S-sequence $f$ in $\mathbb{R}^2$ holds $\mathrm{Rev}(f)$ is special sequence.

(48) For every S-sequence $f$ in $\mathbb{R}^2$ such that $i \in \mathrm{dom}\, f$ holds $f_i \in \widetilde{\mathcal{L}}(f)$.

(49) If $p \neq q$ and if $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\mathcal{L}(p,q)$ is special polygonal arc.

[2] The proposition (43) has been removed.

(50) For every S-sequence $f$ in $\mathbb{R}^2$ such that $p \in \operatorname{rng} f$ and $p \looparrowleft f \neq 1$ holds $f -: p$ is special sequence.

(51) For every S-sequence $f$ in $\mathbb{R}^2$ such that $p \in \operatorname{rng} f$ and $p \looparrowleft f \neq \operatorname{len} f$ holds $f :- p$ is special sequence.

(52) For every S-sequence $f$ in $\mathbb{R}^2$ such that $p \in \mathcal{L}(f, i)$ and $p \notin \operatorname{rng} f$ holds $\operatorname{Ins}(f, i, p)$ is special sequence.

## 3. Special Polygons in $\mathcal{E}^2_{\mathrm{T}}$

Let us observe that there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal arc.
Next we state the proposition

(53) If $P$ is a special polygonal arc joining $p_1$ and $p_2$, then $P$ is a special polygonal arc joining $p_2$ and $p_1$.

Let us consider $p_1$, $p_2$ and let $P$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. We say that $p_1$ and $p_2$ split $P$ if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $p_1 \neq p_2$, and

(ii) there exist S-sequences $f_1$, $f_2$ in $\mathbb{R}^2$ such that $p_1 = (f_1)_1$ and $p_1 = (f_2)_1$ and $p_2 = (f_1)_{\operatorname{len} f_1}$ and $p_2 = (f_2)_{\operatorname{len} f_2}$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{p_1, p_2\}$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$.

We now state four propositions:

(54) If $p_1$ and $p_2$ split $P$, then $p_2$ and $p_1$ split $P$.

(55) If $p_1$ and $p_2$ split $P$ and $q \in P$ and $q \neq p_1$, then $p_1$ and $q$ split $P$.

(56) If $p_1$ and $p_2$ split $P$ and $q \in P$ and $q \neq p_2$, then $q$ and $p_2$ split $P$.

(57) If $p_1$ and $p_2$ split $P$ and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq q_2$, then $q_1$ and $q_2$ split $P$.

Let $P$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Let us observe that $P$ is special polygon if and only if:

(Def. 2) There exist $p_1$, $p_2$ such that $p_1$ and $p_2$ split $P$.

We introduce $P$ is special polygonal as a synonym of $P$ is special polygon.
Let us consider $r_1$, $r_2$, $r'_1$, $r'_2$. The functor $[r_1, r_2, r'_1, r'_2]$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def. 3).

(Def. 3) $[r_1, r_2, r'_1, r'_2] = \{p : p_{\mathbf{1}} = r_1 \wedge p_{\mathbf{2}} \leq r'_2 \wedge p_{\mathbf{2}} \geq r'_1 \vee p_{\mathbf{1}} \leq r_2 \wedge p_{\mathbf{1}} \geq r_1 \wedge p_{\mathbf{2}} = r'_2 \vee p_{\mathbf{1}} \leq r_2 \wedge p_{\mathbf{1}} \geq r_1 \wedge p_{\mathbf{2}} = r'_1 \vee p_{\mathbf{1}} = r_2 \wedge p_{\mathbf{2}} \leq r'_2 \wedge p_{\mathbf{2}} \geq r'_1\}$.

Next we state three propositions:

(58) If $r_1 < r_2$ and $r'_1 < r'_2$, then $[r_1, r_2, r'_1, r'_2] = \mathcal{L}([r_1, r'_1], [r_1, r'_2]) \cup \mathcal{L}([r_1, r'_2], [r_2, r'_2]) \cup (\mathcal{L}([r_2, r'_2], [r_2, r'_1]) \cup \mathcal{L}([r_2, r'_1], [r_1, r'_1]))$.

(59) If $r_1 < r_2$ and $r'_1 < r'_2$, then $[r_1, r_2, r'_1, r'_2]$ is special polygonal.

(60) $\square_{\mathcal{E}^2} = [0, 1, 0, 1]$.

One can check that there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal.
The following proposition is true

(61) $\square_{\mathcal{E}^2}$ is special polygonal.

One can check the following observations:

* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal,

* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal is also non empty, and
* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal is also non trivial.

A special polygon in $\mathbb{R}^2$ is a special polygonal subset of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following propositions:

(62) If $P$ is special polygonal arc, then $P$ is compact.

(63) Every special polygon in $\mathbb{R}^2$ is compact.

(64) If $P$ is special polygonal, then for all $p_1$, $p_2$ such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ holds $p_1$ and $p_2$ split $P$.

(65) Suppose $P$ is special polygonal. Let given $p_1$, $p_2$. Suppose $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$. Then there exist subsets $P_1$, $P_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $P_1$ is a special polygonal arc joining $p_1$ and $p_2$,

(ii) $P_2$ is a special polygonal arc joining $p_1$ and $p_2$,

(iii) $P_1 \cap P_2 = \{p_1, p_2\}$, and

(iv) $P = P_1 \cup P_2$.

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