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Extremal Properties of Vertices on Special Polygons, Part I

Yatsuka Nakamura
Shinshu University
Nagano

Czesław Byliński
Warsaw University
Białystok

Summary. First, extremal properties of endpoints of line segments in n-dimensional Euclidean space are discussed. Some topological properties of line segments are also discussed. Secondly, extremal properties of vertices of special polygons which consist of horizontal and vertical line segments in 2-dimensional Euclidean space, are also derived.

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The articles [14], [18], [3], [15], [11], [2], [16], [6], [13], [8], [1], [4], [12], [5], [7], [17], [9], and [10] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(2)[1] For every finite set A holds A is trivial iff $\mathrm{card} A < 2$.

(3) For every set A holds A is non trivial iff there exist sets a_1, a_2 such that $a_1 \in A$ and $a_2 \in A$ and $a_1 \neq a_2$.

(4) Let D be a set and A be a subset of D. Then A is non trivial if and only if there exist elements d_1, d_2 of D such that $d_1 \in A$ and $d_2 \in A$ and $d_1 \neq d_2$.

We follow the rules: n, i, k, m denote natural numbers and r, r_1, r_2, s, s_1, s_2 denote real numbers. One can prove the following propositions:

(5) If $r \leq s$, then $r - 1 \leq s$ and $r - 1 < s$ and $r \leq s + 1$ and $r < s + 1$.

(6) If $n < k$, then $n \leq k - 1$ and if $r < s$, then $r - 1 \leq s$ and $r - 1 < s$ and $r \leq s + 1$ and $r < s + 1$.

(7) If $1 \leq k - m$ and $k - m \leq n$, then $k - m \in \mathrm{Seg}\, n$ and $k - m$ is a natural number.

(8) If $r_1 \geq 0$ and $r_2 \geq 0$ and $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.

(9) If $r_1 \leq 0$ and $r_2 \leq 0$ and $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.

(10) If $0 \leq r_1$ and $r_1 \leq 1$ and $0 \leq r_2$ and $r_2 \leq 1$ and $r_1 \cdot r_2 = 1$, then $r_1 = 1$ and $r_2 = 1$.

(11) If $r_1 \geq 0$ and $r_2 \geq 0$ and $s_1 \geq 0$ and $s_2 \geq 0$ and $r_1 \cdot s_1 + r_2 \cdot s_2 = 0$, then $r_1 = 0$ or $s_1 = 0$ but $r_2 = 0$ or $s_2 = 0$.

[1] The proposition (1) has been removed.

(12) If $0 \leq r$ and $r \leq 1$ and $s_1 \geq 0$ and $s_2 \geq 0$ and $r \cdot s_1 + (1-r) \cdot s_2 = 0$, then $r = 0$ and $s_2 = 0$ or $r = 1$ and $s_1 = 0$ or $s_1 = 0$ and $s_2 = 0$.

(13) If $r < r_1$ and $r < r_2$, then $r < \min(r_1, r_2)$.

(14) If $r > r_1$ and $r > r_2$, then $r > \max(r_1, r_2)$.

In this article we present several logical schemes. The scheme *FinSeqFam* deals with a non empty set $\mathcal{A}$, a finite sequence $\mathcal{B}$ of elements of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(\mathcal{B}, i) : i \in \operatorname{dom} \mathcal{B} \wedge \mathcal{P}[i]\}$ is finite

for all values of the parameters.

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$\{\mathcal{F}(\mathcal{B}, i) : 1 \leq i \wedge i \leq \operatorname{len} \mathcal{B} \wedge \mathcal{P}[i]\}$ is finite

for all values of the parameters.

Next we state several propositions:

(15) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|x_1 - x_2| - |x_2 - x_3| \leq |x_1 - x_3|$.

(16) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|x_2 - x_1| - |x_2 - x_3| \leq |x_3 - x_1|$.

(17) Every point of $\mathcal{E}^n_{\mathrm{T}}$ is an element of $\mathcal{R}^n$ and a point of $\mathcal{E}^n$.

(18) Every point of $\mathcal{E}^n$ is an element of $\mathcal{R}^n$ and a point of $\mathcal{E}^n_{\mathrm{T}}$.

(19) Every element of $\mathcal{R}^n$ is a point of $\mathcal{E}^n$ and a point of $\mathcal{E}^n_{\mathrm{T}}$.

2. Properties of line segments

We use the following convention: r, s are real numbers and p, p_1, p_2, q_1, q_2 are points of $\mathcal{E}^n_{\mathrm{T}}$.

Next we state a number of propositions:

(20) For all points u_1, u_2 of $\mathcal{E}^n$ and for all elements v_1, v_2 of $\mathcal{R}^n$ such that $v_1 = u_1$ and $v_2 = u_2$ holds $\rho(u_1, u_2) = |v_1 - v_2|$.

(21) For all p, p_1, p_2 such that $p \in \mathcal{L}(p_1, p_2)$ there exists r such that $0 \leq r$ and $r \leq 1$ and $p = (1-r) \cdot p_1 + r \cdot p_2$.

(22) For all p_1, p_2, r such that $0 \leq r$ and $r \leq 1$ holds $(1-r) \cdot p_1 + r \cdot p_2 \in \mathcal{L}(p_1, p_2)$.

(23) Let P be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is closed and $P \subseteq \mathcal{L}(p_1, p_2)$. Then there exists s such that $(1-s) \cdot p_1 + s \cdot p_2 \in P$ and for every r such that $0 \leq r$ and $r \leq 1$ and $(1-r) \cdot p_1 + r \cdot p_2 \in P$ holds $s \leq r$.

(24) For all p_1, p_2, q_1, q_2 such that $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$ and $p_1 \in \mathcal{L}(q_1, q_2)$ holds $p_1 = q_1$ or $p_1 = q_2$.

(25) For all p_1, p_2, q_1, q_2 such that $\mathcal{L}(p_1, p_2) = \mathcal{L}(q_1, q_2)$ holds $p_1 = q_1$ and $p_2 = q_2$ or $p_1 = q_2$ and $p_2 = q_1$.

(26) $\mathcal{E}^n_{\mathrm{T}}$ is a T_2 space.

(27) $\{p\}$ is closed.

(28) $\mathcal{L}(p_1, p_2)$ is compact.

(29) $\mathcal{L}(p_1, p_2)$ is closed.

Let us consider n, p and let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that p is extremal in P if and only if:

(Def. 1) $p \in P$ and for all p_1, p_2 such that $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2) \subseteq P$ holds $p = p_1$ or $p = p_2$.

Next we state several propositions:

(30) For all subsets P, Q of $\mathcal{E}^n_{\mathrm{T}}$ such that p is extremal in P and $Q \subseteq P$ and $p \in Q$ holds p is extremal in Q.

(31) p is extremal in $\{p\}$.

(32) p_1 is extremal in $\mathcal{L}(p_1, p_2)$.

(33) p_2 is extremal in $\mathcal{L}(p_1, p_2)$.

(34) If p is extremal in $\mathcal{L}(p_1, p_2)$, then $p = p_1$ or $p = p_2$.

3. Alternating special sequences

We use the following convention: P, Q are subsets of $\mathcal{E}^2_{\mathrm{T}}$, f, f_1, f_2 are finite sequences of elements of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and p, p_1, p_2, p_3, q are points of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following proposition

(35) For all p_1, p_2 such that $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ there exists p such that $p \in \mathcal{L}(p_1, p_2)$ and $p_{\mathbf{1}} \neq (p_1)_{\mathbf{1}}$ and $p_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (p_1)_{\mathbf{2}}$ and $p_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

Let us consider P. We say that P is horizontal if and only if:

(Def. 2) For all p, q such that $p \in P$ and $q \in P$ holds $p_{\mathbf{2}} = q_{\mathbf{2}}$.

We say that P is vertical if and only if:

(Def. 3) For all p, q such that $p \in P$ and $q \in P$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$.

Let us note that every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non trivial and horizontal is also non vertical and every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non trivial and vertical is also non horizontal.

One can prove the following propositions:

(36) $p_{\mathbf{2}} = q_{\mathbf{2}}$ iff $\mathcal{L}(p, q)$ is horizontal.

(37) $p_{\mathbf{1}} = q_{\mathbf{1}}$ iff $\mathcal{L}(p, q)$ is vertical.

(38) If $p_1 \in \mathcal{L}(p, q)$ and $p_2 \in \mathcal{L}(p, q)$ and $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$, then $\mathcal{L}(p, q)$ is horizontal.

(39) If $p_1 \in \mathcal{L}(p, q)$ and $p_2 \in \mathcal{L}(p, q)$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$, then $\mathcal{L}(p, q)$ is vertical.

(40) $\mathcal{L}(f, i)$ is closed.

(41) If f is special, then $\mathcal{L}(f, i)$ is vertical or $\mathcal{L}(f, i)$ is horizontal.

(42) If f is one-to-one and $1 \leq i$ and $i + 1 \leq \operatorname{len} f$, then $\mathcal{L}(f, i)$ is non trivial.

(43) If f is one-to-one and $1 \leq i$ and $i + 1 \leq \operatorname{len} f$ and $\mathcal{L}(f, i)$ is vertical, then $\mathcal{L}(f, i)$ is non horizontal.

(44) For every f holds $\{\mathcal{L}(f, i) : 1 \leq i \wedge i \leq \operatorname{len} f\}$ is finite.

(45) For every f holds $\{\mathcal{L}(f, i) : 1 \leq i \wedge i + 1 \leq \operatorname{len} f\}$ is finite.

(46) For every f holds $\{\mathcal{L}(f, i) : 1 \leq i \wedge i \leq \operatorname{len} f\}$ is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

(47) For every f holds $\{\mathcal{L}(f, i) : 1 \leq i \wedge i + 1 \leq \operatorname{len} f\}$ is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

(48) For every f such that $Q = \bigcup \{\mathcal{L}(f, i) : 1 \leq i \wedge i + 1 \leq \operatorname{len} f\}$ holds Q is closed.

(49) $\widetilde{\mathcal{L}}(f)$ is closed.

Let I_1 be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. We say that I_1 is alternating if and only if:

(Def. 4) For every i such that $1 \leq i$ and $i+2 \leq \mathrm{len} I_1$ holds $((I_1)_i)_{\mathbf{1}} \neq ((I_1)_{i+2})_{\mathbf{1}}$ and $((I_1)_i)_{\mathbf{2}} \neq ((I_1)_{i+2})_{\mathbf{2}}$.

The following propositions are true:

(50) If f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $(f_i)_{\mathbf{1}} = (f_{i+1})_{\mathbf{1}}$, then $(f_{i+1})_{\mathbf{2}} = (f_{i+2})_{\mathbf{2}}$.

(51) If f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $(f_i)_{\mathbf{2}} = (f_{i+1})_{\mathbf{2}}$, then $(f_{i+1})_{\mathbf{1}} = (f_{i+2})_{\mathbf{1}}$.

(52) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $p_1 = f_i$ and $p_2 = f_{i+1}$ and $p_3 = f_{i+2}$. Then $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

(53) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $p_1 = f_i$ and $p_2 = f_{i+1}$ and $p_3 = f_{i+2}$. Then $(p_2)_{\mathbf{1}} = (p_3)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ or $(p_2)_{\mathbf{2}} = (p_3)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

(54) If f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$, then $\mathcal{L}(f_i, f_{i+2}) \not\subseteq \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$.

(55) If f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $\mathcal{L}(f,i)$ is vertical, then $\mathcal{L}(f,i+1)$ is horizontal.

(56) If f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $\mathcal{L}(f,i)$ is horizontal, then $\mathcal{L}(f,i+1)$ is vertical.

(57) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$. Then $\mathcal{L}(f,i)$ is vertical and $\mathcal{L}(f,i+1)$ is horizontal or $\mathcal{L}(f,i)$ is horizontal and $\mathcal{L}(f,i+1)$ is vertical.

(58) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $f_{i+1} \in \mathcal{L}(p,q)$ and $\mathcal{L}(p,q) \subseteq \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$. Then $f_{i+1} = p$ or $f_{i+1} = q$.

(59) If f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$, then f_{i+1} is extremal in $\mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$.

(60) Let u be a point of $\mathcal{E}^2$. Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \mathrm{len} f$ and $u = f_{i+1}$ and $f_{i+1} \in \mathcal{L}(p,q)$ and $f_{i+1} \neq q$ and $p \notin \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$. Let given s. If $s > 0$, then there exists p_3 such that $p_3 \notin \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$ and $p_3 \in \mathcal{L}(p,q)$ and $p_3 \in \mathrm{Ball}(u,s)$.

Let us consider f_1, f_2, P. We say that f_1 and f_2 are generators of P if and only if the conditions (Def. 5) are satisfied.

(Def. 5) f_1 is alternating and special sequence and f_2 is alternating and special sequence and $(f_1)_1 = (f_2)_1$ and $(f_1)_{\mathrm{len} f_1} = (f_2)_{\mathrm{len} f_2}$ and $\langle (f_1)_2, (f_1)_1, (f_2)_2 \rangle$ is alternating and $\langle (f_1)_{\mathrm{len} f_1 - 1}, (f_1)_{\mathrm{len} f_1}, (f_2)_{\mathrm{len} f_2 - 1} \rangle$ is alternating and $(f_1)_1 \neq (f_1)_{\mathrm{len} f_1}$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{(f_1)_1, (f_1)_{\mathrm{len} f_1}\}$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$.

We now state the proposition

(61) If f_1 and f_2 are generators of P and $1 < i$ and $i < \mathrm{len} f_1$, then $(f_1)_i$ is extremal in P.

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