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The Formalization of Simple Graphs

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Summary. A graph is simple when

- it is non-directed,
- there is at most one edge between two vertices,
- there is no loop of length one.

A formalization of simple graphs is given from scratch. There is already an article [10], dealing with the similar subject. It is not used as a starting-point, because [10] formalizes directed non-empty graphs. Given a set of vertices, edge is defined as an (unordered) pair of different two vertices and graph as a pair of a set of vertices and a set of edges.

The following concepts are introduced:

- simple graph structure,
- the set of all simple graphs,
- equality relation on graphs.
- the notion of degrees of vertices; the number of edges connected to, or the number of adjacent vertices,
- the notion of subgraphs,
- path, cycle,
- complete and bipartite complete graphs,

Theorems proved in this articles include:

- the set of simple graphs satisfies a certain minimality condition,
- equivalence between two notions of degrees.

MML Identifier: `SGRAPH1`.

WWW: `http://mizar.org/JFM/Vol6/sgraph1.html`

The articles [12], [7], [15], [13], [2], [1], [4], [5], [6], [3], [9], [8], [14], and [11] provide the notation and terminology for this paper.

1. Preliminaries

Let m, n be natural numbers. The functor $[m,n]_{\mathbb{N}}$ yields a subset of $\mathbb{N}$ and is defined by:

(Def. 1) $[m,n]_{\mathbb{N}} = \{i; i$ ranges over natural numbers: $m \leq i \wedge i \leq n\}$.

Let m, n be natural numbers. Observe that $[m,n]_{\mathbb{N}}$ is finite.

Next we state several propositions:

(2)[1] Let m, n be natural numbers and e be a set. Then $e \in [m,n]_{\mathbb{N}}$ if and only if there exists a natural number i such that $e = i$ and $m \leq i$ and $i \leq n$.

(3) For all natural numbers m, n, k holds $k \in [m,n]_{\mathbb{N}}$ iff $m \leq k$ and $k \leq n$.

[1] The proposition (1) has been removed.

(4) For every natural number n holds $[1,n]_{\mathbb{N}} = \operatorname{Seg} n$.

(5) For all natural numbers m, n such that $1 \leq m$ holds $[m,n]_{\mathbb{N}} \subseteq \operatorname{Seg} n$.

(6) For all natural numbers k, m, n such that $k < m$ holds $\operatorname{Seg} k$ misses $[m,n]_{\mathbb{N}}$.

(7) For all natural numbers m, n such that $n < m$ holds $[m,n]_{\mathbb{N}} = \emptyset$.

Let A be a set. The functor TwoElementSets(A) yields a set and is defined as follows:

(Def. 4)[2] TwoElementSets$(A) = \{z; z$ ranges over finite elements of 2^A: $\operatorname{card} z = 2\}$.

One can prove the following propositions:

(9)[3] For every set A and for every set e holds $e \in$ TwoElementSets(A) iff there exists a finite subset z of A such that $e = z$ and $\operatorname{card} z = 2$.

(10) Let A be a set and e be a set. Then $e \in$ TwoElementSets(A) if and only if the following conditions are satisfied:

(i) e is a finite subset of A, and

(ii) there exist sets x, y such that $x \in A$ and $y \in A$ and $x \neq y$ and $e = \{x, y\}$.

(11) For every set A holds TwoElementSets$(A) \subseteq 2^A$.

(12) For every set A and for all sets e_1, e_2 such that $\{e_1, e_2\} \in$ TwoElementSets(A) holds $e_1 \in A$ and $e_2 \in A$ and $e_1 \neq e_2$.

(13) TwoElementSets$(\emptyset) = \emptyset$.

(14) For all sets t, u such that $t \subseteq u$ holds TwoElementSets$(t) \subseteq$ TwoElementSets(u).

(15) For every finite set A holds TwoElementSets(A) is finite.

(16) For every non trivial set A holds TwoElementSets(A) is non empty.

(17) For every set a holds TwoElementSets$(\{a\}) = \emptyset$.

Let X be an empty set. Observe that every subset of X is empty.
In the sequel X denotes a set.

2. Simple Graphs

We consider simple graph structures as extensions of 1-sorted structure as systems
⟨ a carrier, SEdges ⟩,
where the carrier is a set and the SEdges constitute a subset of TwoElementSets(the carrier).

Let X be a set. The functor SimpleGraphs(X) yields a set and is defined as follows:

(Def. 6)[4] SimpleGraphs$(X) = \{\langle v, e \rangle : v$ ranges over finite subsets of X, e ranges over finite subsets of TwoElementSets$(v)\}$.

We now state the proposition

(19)[5] $\langle \emptyset, \emptyset_{\mathrm{TwoElementSets}(\emptyset)} \rangle \in$ SimpleGraphs(X).

Let X be a set. Observe that SimpleGraphs(X) is non empty.
Let X be a set. A strict simple graph structure is said to be a simple graph of X if:

(Def. 7) It is an element of SimpleGraphs(X).

The following proposition is true

(21)[6] Let g be a set. Then $g \in$ SimpleGraphs(X) if and only if there exists a finite subset v of X and there exists a finite subset e of TwoElementSets(v) such that $g = \langle v, e \rangle$.

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[2] The definitions (Def. 2) and (Def. 3) have been removed.
[3] The proposition (8) has been removed.
[4] The definition (Def. 5) has been removed.
[5] The proposition (18) has been removed.
[6] The proposition (20) has been removed.

3. Equality Relation on Simple Graphs

We now state four propositions:

(23)[7] For every simple graph g of X holds the carrier of $g \subseteq X$ and the SEdges of $g \subseteq$ TwoElementSets(the carrier of g).

(25)[8] Let g be a simple graph of X and e be a set. Suppose $e \in$ the SEdges of g. Then there exist sets v_1, v_2 such that $v_1 \in$ the carrier of g and $v_2 \in$ the carrier of g and $v_1 \neq v_2$ and $e = \{v_1, v_2\}$.

(26) Let g be a simple graph of X and v_1, v_2 be sets. Suppose $\{v_1, v_2\} \in$ the SEdges of g. Then $v_1 \in$ the carrier of g and $v_2 \in$ the carrier of g and $v_1 \neq v_2$.

(27) Let g be a simple graph of X. Then
 (i) the carrier of g is a finite subset of X, and
 (ii) the SEdges of g are a finite subset of TwoElementSets(the carrier of g).

Let us consider X and let G, G' be simple graphs of X. We say that G is isomorphic to G' if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exists a function F_1 from the carrier of G into the carrier of G' such that
 (i) F_1 is bijective, and
 (ii) for all elements v_1, v_2 of G holds $\{v_1, v_2\} \in$ the SEdges of G iff $\{F_1(v_1), F_1(v_2)\} \in$ the SEdges of G.

4. Properties of Simple Graphs

The scheme *IndSimpleGraphs0* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:
For every set G such that $G \in$ SimpleGraphs($\mathcal{A}$) holds $\mathcal{P}[G]$
provided the following conditions are met:

- $\mathcal{P}[\langle \emptyset, \emptyset_{\mathrm{TwoElementSets}(\emptyset)} \rangle]$,
- Let g be a simple graph of $\mathcal{A}$ and v be a set. Suppose $g \in$ SimpleGraphs($\mathcal{A}$) and $\mathcal{P}[g]$ and $v \in \mathcal{A}$ and $v \notin$ the carrier of g. Then $\mathcal{P}[\langle (\text{the carrier of } g) \cup \{v\}, \emptyset_{\mathrm{TwoElementSets}((\text{the carrier of } g) \cup \{v\})} \rangle]$, and
- Let g be a simple graph of $\mathcal{A}$ and e be a set. Suppose $\mathcal{P}[g]$ and $e \in$ TwoElementSets(the carrier of g) and $e \notin$ the SEdges of g. Then there exists a subset s_1 of TwoElementSets(the carrier of g) such that $s_1 =$ (the SEdges of g) $\cup \{e\}$ and $\mathcal{P}[\langle$the carrier of g, $s_1\rangle]$.

We now state three propositions:

(28) Let g be a simple graph of X. Then $g = \langle \emptyset, \emptyset_{\mathrm{TwoElementSets}(\emptyset)} \rangle$ or there exists a set v and there exists a subset e of TwoElementSets(v) such that v is non empty and $g = \langle v, e \rangle$.

(30)[9] Let V be a subset of X, E be a subset of TwoElementSets(V), n be a set, and E_1 be a finite subset of TwoElementSets($V \cup \{n\}$). If $\langle V, E \rangle \in$ SimpleGraphs(X) and $n \in X$ and $n \notin V$, then $\langle V \cup \{n\}, E_1 \rangle \in$ SimpleGraphs(X).

(31) Let V be a subset of X, E be a subset of TwoElementSets(V), and v_1, v_2 be sets. Suppose $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\langle V, E \rangle \in$ SimpleGraphs(X). Then there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in$ SimpleGraphs(X).

Let X be a set and let G_1 be a set. We say that G_1 is a set of simple graphs of X if and only if the conditions (Def. 9) are satisfied.

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[7] The proposition (22) has been removed.
[8] The proposition (24) has been removed.
[9] The proposition (29) has been removed.

(Def. 9)(i) $\langle \emptyset, \emptyset_{\mathrm{TwoElementSets}(\emptyset)} \rangle \in G_1$,

(ii) for every subset V of X and for every subset E of TwoElementSets(V) and for every set n and for every finite subset E_1 of TwoElementSets($V \cup \{n\}$) such that $\langle V, E \rangle \in G_1$ and $n \in X$ and $n \notin V$ holds $\langle V \cup \{n\}, E_1 \rangle \in G_1$, and

(iii) for every subset V of X and for every subset E of TwoElementSets(V) and for all sets v_1, v_2 such that $\langle V, E \rangle \in G_1$ and $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\{v_1, v_2\} \notin E$ there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in G_1$.

One can prove the following propositions:

(35)[10] SimpleGraphs(X) is a set of simple graphs of X.

(36) For every set O_1 such that O_1 is a set of simple graphs of X holds SimpleGraphs(X) $\subseteq O_1$.

(37) SimpleGraphs(X) is a set of simple graphs of X and for every set O_1 such that O_1 is a set of simple graphs of X holds SimpleGraphs(X) $\subseteq O_1$.

5. Subgraphs

Let X be a set and let G be a simple graph of X. A simple graph of X is said to be a subgraph of G if:

(Def. 10) The carrier of it $\subseteq$ the carrier of G and the SEdges of it $\subseteq$ the SEdges of G.

6. Degree of Vertices

Let X be a set, let G be a simple graph of X, and let v be a set. The functor degree(G, v) yields a natural number and is defined by:

(Def. 11) There exists a finite set X such that for every set z holds $z \in X$ iff $z \in$ the SEdges of G and $v \in z$ and degree(G, v) $=$ card X.

The following propositions are true:

(39)[11] Let X be a non empty set, G be a simple graph of X, and v be a set. Then there exists a finite set w_1 such that $w_1 = \{w; w$ ranges over elements of X: $w \in$ the carrier of $G \wedge \{v, w\} \in$ the SEdges of $G\}$ and degree(G, v) $=$ card w_1.

(40) Let X be a non empty set, g be a simple graph of X, and v be a set. Suppose $v \in$ the carrier of g. Then there exists a finite set V_1 such that $V_1 =$ the carrier of g and degree(g, v) $<$ card V_1.

(41) Let g be a simple graph of X and v, e be sets. If $v \in$ the carrier of g and $e \in$ the SEdges of g and degree(g, v) $= 0$, then $v \notin e$.

(42) Let g be a simple graph of X, v be a set, and v_4 be a finite set. Suppose $v_4 =$ the carrier of g and $v \in v_4$ and $1 +$ degree(g, v) $=$ card v_4. Let w be an element of v_4. If $v \neq w$, then there exists a set e such that $e \in$ the SEdges of g and $e = \{v, w\}$.

7. Path and Cycle

Let X be a set, let g be a simple graph of X, let v_1, v_2 be elements of g, and let p be a finite sequence of elements of the carrier of g. We say that p is a path of v_1 and v_2 if and only if the conditions (Def. 12) are satisfied.

[10] The propositions (32)–(34) have been removed.

[11] The proposition (38) has been removed.

(Def. 12)(i) $p(1) = v_1$,

(ii) $p(\operatorname{len} p) = v_2$,

(iii) for every natural number i such that $1 \leq i$ and $i < \operatorname{len} p$ holds $\{p(i), p(i+1)\} \in$ the SEdges of g, and

(iv) for all natural numbers i, j such that $1 \leq i$ and $i < \operatorname{len} p$ and $i < j$ and $j < \operatorname{len} p$ holds $p(i) \neq p(j)$ and $\{p(i), p(i+1)\} \neq \{p(j), p(j+1)\}$.

Let X be a set, let g be a simple graph of X, and let v_1, v_2 be elements of the carrier of g. The functor $\mathrm{Paths}(v_1, v_2)$ yields a subset of (the carrier of g)* and is defined as follows:

(Def. 13) $\mathrm{Paths}(v_1, v_2) = \{s_2; s_2$ ranges over elements of (the carrier of g)*: s_2 is a path of v_1 and $v_2\}$.

The following two propositions are true:

(44)[12] Let g be a simple graph of X, v_1, v_2 be elements of the carrier of g, and e be a set. Then $e \in \mathrm{Paths}(v_1, v_2)$ if and only if there exists an element s_2 of (the carrier of g)* such that $e = s_2$ and s_2 is a path of v_1 and v_2.

(45) Let g be a simple graph of X, v_1, v_2 be elements of the carrier of g, and e be an element of (the carrier of g)*. If e is a path of v_1 and v_2, then $e \in \mathrm{Paths}(v_1, v_2)$.

Let X be a set, let g be a simple graph of X, and let p be a set. We say that p is a cycle of g if and only if:

(Def. 14) There exists an element v of the carrier of g such that $p \in \mathrm{Paths}(v, v)$.

8. Some Famous Graphs

Let n, m be natural numbers. The functor $\mathrm{K}_{m,n}$ yielding a simple graph of $\mathbb{N}$ is defined by the condition (Def. 16).

(Def. 16)[13] There exists a subset e_3 of $\mathrm{TwoElementSets}(\mathrm{Seg}(m+n))$ such that $e_3 = \{\{i, j\}; i$ ranges over elements of $\mathbb{N}$, j ranges over elements of $\mathbb{N}$: $i \in \mathrm{Seg}\, m \wedge j \in [m+1, m+n]_{\mathbb{N}}\}$ and $\mathrm{K}_{m,n} = \langle \mathrm{Seg}(m+n), e_3 \rangle$.

Let n be a natural number. The functor K_n yielding a simple graph of $\mathbb{N}$ is defined by the condition (Def. 17).

(Def. 17) There exists a finite subset e_3 of $\mathrm{TwoElementSets}(\mathrm{Seg}\, n)$ such that $e_3 = \{\{i, j\}; i$ ranges over elements of $\mathbb{N}$, j ranges over elements of $\mathbb{N}$: $i \in \mathrm{Seg}\, n \wedge j \in \mathrm{Seg}\, n \wedge i \neq j\}$ and $\mathrm{K}_n = \langle \mathrm{Seg}\, n, e_3 \rangle$.

The simple graph TriangleGraph of $\mathbb{N}$ is defined by:

(Def. 18) $\mathrm{TriangleGraph} = \mathrm{K}_3$.

The following propositions are true:

(46) There exists a subset e_3 of $\mathrm{TwoElementSets}(\mathrm{Seg}\, 3)$ such that $e_3 = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ and $\mathrm{TriangleGraph} = \langle \mathrm{Seg}\, 3, e_3 \rangle$.

(47) The carrier of $\mathrm{TriangleGraph} = \mathrm{Seg}\, 3$ and the SEdges of $\mathrm{TriangleGraph} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$.

(48) $\{1, 2\} \in$ the SEdges of TriangleGraph and $\{2, 3\} \in$ the SEdges of TriangleGraph and $\{3, 1\} \in$ the SEdges of TriangleGraph.

(49) $\langle 1 \rangle \frown \langle 2 \rangle \frown \langle 3 \rangle \frown \langle 1 \rangle$ is a cycle of TriangleGraph.

[12] The proposition (43) has been removed.

[13] The definition (Def. 15) has been removed.

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