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Semi-Affine Space[1]

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Summary. A brief survey on semi-affine geometry, which results from the classical Pappian and Desarguesian affine (dimension free) geometry by weakening the so called trapezium axiom. With the help of the relation of parallelogram in every semi-affine space we define the operation of "addition" of "vectors". Next we investigate in greater details the relation of (affine) trapezium in such spaces.

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The article [1] provides the notation and terminology for this paper.

Let I_1 be a non empty affine structure. We say that I_1 is semi affine space-like if and only if the conditions (Def. 1) are satisfied.

(Def. 1) For all elements a, b of I_1 holds $a,b \upuparrows b,a$ and for all elements a, b, c of I_1 holds $a,b \upuparrows c,c$ and for all elements a, b, p, q, r, s of I_1 such that $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$ holds $p,q \upuparrows r,s$ and for all elements a, b, c of I_1 such that $a,b \upuparrows a,c$ holds $b,a \upuparrows b,c$ and there exist elements a, b, c of I_1 such that $a,b \not\upuparrows a,c$ and for all elements a, b, p of I_1 there exists an element q of I_1 such that $a,b \upuparrows p,q$ and $a,p \upuparrows b,q$ and for all elements o, a of I_1 there exists an element p of I_1 such that for all elements b, c of I_1 holds $o,a \upuparrows o,p$ and there exists an element d of I_1 such that if $o,p \upuparrows o,b$, then $o,c \upuparrows o,d$ and $p,c \upuparrows b,d$ and for all elements o, a, a', b, b', c, c' of I_1 such that $o,a \not\upuparrows o,b$ and $o,a \not\upuparrows o,c$ and $o,a \upuparrows o,a'$ and $o,b \upuparrows o,b'$ and $o,c \upuparrows o,c'$ and $a,b \upuparrows a',b'$ and $a,c \upuparrows a',c'$ holds $b,c \upuparrows b',c'$ and for all elements a, a', b, b', c, c' of I_1 such that $a,a' \not\upuparrows a,b$ and $a,a' \not\upuparrows a,c$ and $a,a' \upuparrows b,b'$ and $a,a' \upuparrows c,c'$ and $a,b \upuparrows a',b'$ and $a,c \upuparrows a',c'$ holds $b,c \upuparrows b',c'$ and for all elements a_1, a_2, a_3, b_1, b_2, b_3 of I_1 such that $a_1,a_2 \upuparrows a_1,a_3$ and $b_1,b_2 \upuparrows b_1,b_3$ and $a_1,b_2 \upuparrows a_2,b_1$ and $a_2,b_3 \upuparrows a_3,b_2$ holds $a_3,b_1 \upuparrows a_1,b_3$ and for all elements a, b, c, d of I_1 such that $a,b \not\upuparrows a,c$ and $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ holds $a,d \not\upuparrows b,c$.

Let us observe that there exists a non empty affine structure which is semi affine space-like.

A semi affine space is a semi affine space-like non empty affine structure.

We use the following convention: S_1 denotes a semi affine space and a, a', a_1, a_2, a_3, a_4, b, b', c, c', d, d', d_1, d_2, o, p, p_1, p_2, q, r, r_1, r_2, s, x, y, z denote elements of S_1.

The following propositions are true:

(12)[1] $a,b \upuparrows a,b$.

(13) If $a,b \upuparrows c,d$, then $c,d \upuparrows a,b$.

(14) $a,a \upuparrows b,c$.

[1]Supported by RPBP.III-24.C2.
[1] The propositions (1)–(11) have been removed.

(15) If $a,b \parallel c,d$, then $b,a \parallel c,d$.

(16) If $a,b \parallel c,d$, then $a,b \parallel d,c$.

(17) If $a,b \parallel c,d$, then $b,a \parallel c,d$ and $a,b \parallel d,c$ and $b,a \parallel d,c$ and $c,d \parallel a,b$ and $d,c \parallel a,b$ and $c,d \parallel b,a$ and $d,c \parallel b,a$.

(18) Suppose $a,b \parallel a,c$. Then $a,c \parallel a,b$ and $b,a \parallel a,c$ and $a,b \parallel c,a$ and $a,c \parallel b,a$ and $b,a \parallel c,a$ and $c,a \parallel a,b$ and $c,a \parallel b,a$ and $b,a \parallel b,c$ and $a,b \parallel b,c$ and $b,a \parallel c,b$ and $b,c \parallel b,a$ and $a,b \parallel c,b$ and $c,b \parallel b,a$ and $b,c \parallel a,b$ and $c,b \parallel a,b$ and $c,a \parallel c,b$ and $a,c \parallel c,b$ and $c,a \parallel b,c$ and $a,c \parallel b,c$ and $c,b \parallel c,a$ and $b,c \parallel c,a$ and $c,b \parallel a,c$ and $b,c \parallel a,c$.

(20)[2] If $a \neq b$ and $p,q \parallel a,b$ and $a,b \parallel r,s$, then $p,q \parallel r,s$.

(21) If $a,b \nparallel a,d$, then $a \neq b$ and $b \neq d$ and $d \neq a$.

(22) If $a,b \nparallel p,q$, then $a \neq b$ and $p \neq q$.

(23) If $a,b \parallel a,x$ and $b,c \parallel b,x$ and $c,a \parallel c,x$, then $a,b \parallel a,c$.

(25)[3] If $a,b \nparallel a,c$ and $p \neq q$, then $p,q \nparallel p,a$ or $p,q \nparallel p,b$ or $p,q \nparallel p,c$.

(26) If $p \neq q$, then there exists r such that $p,q \nparallel p,r$.

(28)[4] Suppose $a,b \nparallel a,c$. Then $a,b \nparallel c,a$ and $b,a \nparallel a,c$ and $b,a \nparallel c,a$ and $a,c \nparallel a,b$ and $a,c \nparallel b,a$ and $c,a \nparallel a,b$ and $c,a \nparallel b,a$ and $b,a \nparallel b,c$ and $b,a \nparallel c,b$ and $a,b \nparallel b,c$ and $a,b \nparallel c,b$ and $b,c \nparallel b,a$ and $b,c \nparallel a,b$ and $c,b \nparallel a,b$ and $c,b \nparallel b,a$ and $c,b \nparallel c,a$ and $c,b \nparallel a,c$ and $b,c \nparallel c,a$ and $b,c \nparallel a,c$ and $c,a \nparallel c,b$ and $c,a \nparallel b,c$ and $a,c \nparallel b,c$ and $a,c \nparallel c,b$.

(29) If $a,b \nparallel c,d$ and $a,b \parallel p,q$ and $c,d \parallel r,s$ and $p \neq q$ and $r \neq s$, then $p,q \nparallel r,s$.

(30) If $a,b \nparallel a,c$ and $a,b \parallel p,q$ and $a,c \parallel p,r$ and $b,c \parallel q,r$ and $p \neq q$, then $p,q \nparallel p,r$.

(31) If $a,b \nparallel a,c$ and $a,c \parallel p,r$ and $b,c \parallel p,r$, then $p = r$.

(32) If $p,q \nparallel p,r_1$ and $p,r_1 \parallel p,r_2$ and $q,r_1 \parallel q,r_2$, then $r_1 = r_2$.

(33) If $a,b \nparallel a,c$ and $a,b \parallel p,q$ and $a,c \parallel p,r_1$ and $a,c \parallel p,r_2$ and $b,c \parallel q,r_1$ and $b,c \parallel q,r_2$, then $r_1 = r_2$.

(34) If $a = b$ or $c = d$ or $a = c$ and $b = d$ or $a = d$ and $b = c$, then $a,b \parallel c,d$.

(35) If $a = b$ or $a = c$ or $b = c$, then $a,b \parallel a,c$.

Let us consider S_1, a, b, c. We say that a, b and c are collinear if and only if:

(Def. 2) $a,b \parallel a,c$.

We now state a number of propositions:

(37)[5] Suppose a_1, a_2 and a_3 are collinear. Then

(i) a_1, a_3 and a_2 are collinear,

(ii) a_2, a_1 and a_3 are collinear,

(iii) a_2, a_3 and a_1 are collinear,

(iv) a_3, a_1 and a_2 are collinear, and

(v) a_3, a_2 and a_1 are collinear.

[2] The proposition (19) has been removed.
[3] The proposition (24) has been removed.
[4] The proposition (27) has been removed.
[5] The proposition (36) has been removed.

(39)[6] If a, b and c are not collinear and $a,b \upuparrows p,q$ and $a,c \upuparrows p,r$ and $p \neq q$ and $p \neq r$, then p, q and r are not collinear.

(40) If $a = b$ or $b = c$ or $c = a$, then a, b and c are collinear.

(41) If $p \neq q$, then there exists r such that p, q and r are not collinear.

(42) If a, b and c are collinear and a, b and d are collinear, then $a,b \upuparrows c,d$.

(43) If a, b and c are not collinear and $a,b \upuparrows c,d$, then a, b and d are not collinear.

(44) Suppose a, b and c are not collinear and $a,b \upuparrows c,d$ and $c \neq d$ and c, d and x are collinear. Then a, b and x are not collinear.

(45) If o, a and b are not collinear and o, a and x are collinear and o, b and x are collinear, then $o = x$.

(46) Suppose $o \neq a$ and $o \neq b$ and o, a and b are collinear and o, a and a' are collinear and o, b and b' are collinear. Then $a,b \upuparrows a',b'$.

(48)[7] Suppose that

(i) $a,b \not\upuparrows c,d$,

(ii) a, b and p_1 are collinear,

(iii) a, b and p_2 are collinear,

(iv) c, d and p_1 are collinear, and

(v) c, d and p_2 are collinear.

Then $p_1 = p_2$.

(49) If $a \neq b$ and a, b and c are collinear and $a,b \upuparrows c,d$, then $a,c \upuparrows b,d$.

(50) If $a \neq b$ and a, b and c are collinear and $a,b \upuparrows c,d$, then $c,b \upuparrows c,d$.

(51) Suppose that o, a and c are not collinear and o, a and b are collinear and o, c and d_1 are collinear and o, c and d_2 are collinear and $a,c \upuparrows b,d_1$ and $a,c \upuparrows b,d_1$ and $a,c \upuparrows b,d_2$. Then $d_1 = d_2$.

(52) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

Let us consider S_1, a, b, c, d. We say that a, b, c, d form a parallelogram if and only if:

(Def. 3) a, b and c are not collinear and $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$.

The following propositions are true:

(54)[8] If a, b, c, d form a parallelogram, then $a \neq b$ and $a \neq c$ and $c \neq b$ and $a \neq d$ and $b \neq d$ and $c \neq d$.

(55) Suppose a, b, c, d form a parallelogram. Then

(i) a, b and c are not collinear,

(ii) b, a and d are not collinear,

(iii) c, d and a are not collinear, and

(iv) d, c and b are not collinear.

[6] The proposition (38) has been removed.
[7] The proposition (47) has been removed.
[8] The proposition (53) has been removed.

(56) Suppose a_1, a_2, a_3, a_4 form a parallelogram. Then a_1, a_2 and a_3 are not collinear and a_1, a_3 and a_2 are not collinear and a_1, a_2 and a_4 are not collinear and a_1, a_4 and a_2 are not collinear and a_1, a_3 and a_4 are not collinear and a_1, a_4 and a_3 are not collinear and a_2, a_1 and a_3 are not collinear and a_2, a_3 and a_1 are not collinear and a_2, a_1 and a_4 are not collinear and a_2, a_4 and a_1 are not collinear and a_2, a_3 and a_4 are not collinear and a_2, a_4 and a_3 are not collinear and a_3, a_1 and a_2 are not collinear and a_3, a_2 and a_1 are not collinear and a_3, a_1 and a_4 are not collinear and a_3, a_4 and a_1 are not collinear and a_3, a_2 and a_4 are not collinear and a_3, a_4 and a_2 are not collinear and a_4, a_1 and a_2 are not collinear and a_4, a_2 and a_1 are not collinear and a_4, a_1 and a_3 are not collinear and a_4, a_3 and a_1 are not collinear and a_4, a_2 and a_3 are not collinear and a_4, a_3 and a_2 are not collinear.

(57) If a, b, c, d form a parallelogram, then a, b and x are not collinear or c, d and x are not collinear.

(58) If a, b, c, d form a parallelogram, then a, c, b, d form a parallelogram.

(59) If a, b, c, d form a parallelogram, then c, d, a, b form a parallelogram.

(60) If a, b, c, d form a parallelogram, then b, a, d, c form a parallelogram.

(61) Suppose a, b, c, d form a parallelogram. Then

(i) a, c, b, d form a parallelogram,

(ii) c, d, a, b form a parallelogram,

(iii) b, a, d, c form a parallelogram,

(iv) c, a, d, b form a parallelogram,

(v) d, b, c, a form a parallelogram, and

(vi) b, d, a, c form a parallelogram.

(62) If a, b and c are not collinear, then there exists d such that a, b, c, d form a parallelogram.

(63) If a, b, c, d_1 form a parallelogram and a, b, c, d_2 form a parallelogram, then $d_1 = d_2$.

(64) If a, b, c, d form a parallelogram, then $a,d \not\upharpoonleft\!\upharpoonright b,c$.

(65) If a, b, c, d form a parallelogram, then a, b, d, c do not form a parallelogram.

(66) If $a \neq b$, then there exists c such that a, b and c are collinear and $c \neq a$ and $c \neq b$.

(67) If a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then $b,c \upharpoonleft\!\upharpoonright b',c'$.

(68) Suppose b, b' and c are not collinear and a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram. Then b, b', c, c' form a parallelogram.

(69) Suppose that

(i) a, b and c are collinear,

(ii) $b \neq c$,

(iii) a, a', b, b' form a parallelogram, and

(iv) a, a', c, c' form a parallelogram.

Then b, b', c, c' form a parallelogram.

(70) Suppose that

(i) a, a', b, b' form a parallelogram,

(ii) a, a', c, c' form a parallelogram, and

(iii) b, b', d, d' form a parallelogram.

Then $c,d \upharpoonleft\!\upharpoonright c',d'$.

(71) If $a \neq d$, then there exist b, c such that a, b, c, d form a parallelogram.

Let us consider S_1, a, b, r, s. We say that a, b are congruent to r, s if and only if:

(Def. 4) $a = b$ and $r = s$ or there exist p, q such that p, q, a, b form a parallelogram and p, q, r, s form a parallelogram.

Next we state a number of propositions:

(73)[9] If a, a are congruent to b, c, then $b = c$.

(74) If a, b are congruent to c, c, then $a = b$.

(75) If a, b are congruent to b, a, then $a = b$.

(76) If a, b are congruent to c, d, then $a,b \upuparrows c,d$.

(77) If a, b are congruent to c, d, then $a,c \upuparrows b,d$.

(78) If a, b are congruent to c, d and a, b and c are not collinear, then a, b, c, d form a parallelogram.

(79) If a, b, c, d form a parallelogram, then a, b are congruent to c, d.

(80) Suppose a, b are congruent to c, d and a, b and c are collinear and r, s, a, b form a parallelogram. Then r, s, c, d form a parallelogram.

(81) If a, b are congruent to c, x and a, b are congruent to c, y, then $x = y$.

(82) There exists d such that a, b are congruent to c, d.

(84)[10] a, b are congruent to a, b.

(85) If r, s are congruent to a, b and r, s are congruent to c, d, then a, b are congruent to c, d.

(86) If a, b are congruent to c, d, then c, d are congruent to a, b.

(87) If a, b are congruent to c, d, then b, a are congruent to d, c.

(88) If a, b are congruent to c, d, then a, c are congruent to b, d.

(89) Suppose a, b are congruent to c, d. Then c, d are congruent to a, b and b, a are congruent to d, c and a, c are congruent to b, d and d, c are congruent to b, a and b, d are congruent to a, c and c, a are congruent to d, b and d, b are congruent to c, a.

(90) If a, b are congruent to p, q and b, c are congruent to q, s, then a, c are congruent to p, s.

(91) If b, a are congruent to p, q and c, a are congruent to p, r, then b, c are congruent to r, q.

(92) If a, o are congruent to o, p and b, o are congruent to o, q, then a, b are congruent to q, p.

(93) If b, a are congruent to p, q and c, a are congruent to p, r, then $b,c \upuparrows q,r$.

(94) If a, o are congruent to o, p and b, o are congruent to o, q, then $a,b \upuparrows p,q$.

Let us consider S_1, a, b, o. The functor $\mathrm{sum}_o(a,b)$ yielding an element of S_1 is defined as follows:

(Def. 5) o, a are congruent to b, $\mathrm{sum}_o(a,b)$.

Let us consider S_1, a, o. The functor $\mathrm{opposite}_o(a)$ yielding an element of S_1 is defined by:

(Def. 6) $\mathrm{sum}_o(a, \mathrm{opposite}_o(a)) = o$.

[9] The proposition (72) has been removed.
[10] The proposition (83) has been removed.

Let us consider S_1, a, b, o. The functor $\mathrm{diff}_o(a,b)$ yielding an element of S_1 is defined as follows:

(Def. 7) $\mathrm{diff}_o(a,b) = \mathrm{sum}_o(a, \mathrm{opposite}_o(b))$.

The following propositions are true:

(99)[11] $\mathrm{sum}_o(a,o) = a$.

(100) There exists x such that $\mathrm{sum}_o(a,x) = o$.

(101) $\mathrm{sum}_o(\mathrm{sum}_o(a,b),c) = \mathrm{sum}_o(a,\mathrm{sum}_o(b,c))$.

(102) $\mathrm{sum}_o(a,b) = \mathrm{sum}_o(b,a)$.

(103) If $\mathrm{sum}_o(a,a) = o$, then $a = o$.

(104) If $\mathrm{sum}_o(a,x) = \mathrm{sum}_o(a,y)$, then $x = y$.

(106)[12] a, o are congruent to o, $\mathrm{opposite}_o(a)$.

(107) If $\mathrm{opposite}_o(a) = \mathrm{opposite}_o(b)$, then $a = b$.

(108) $a,b \upharpoonleft\!\upharpoonright \mathrm{opposite}_o(a), \mathrm{opposite}_o(b)$.

(109) $\mathrm{opposite}_o(o) = o$.

(110) $p,q \upharpoonleft\!\upharpoonright \mathrm{sum}_o(p,r), \mathrm{sum}_o(q,r)$.

(111) If $p,q \upharpoonleft\!\upharpoonright r,s$, then $p,q \upharpoonleft\!\upharpoonright \mathrm{sum}_o(p,r), \mathrm{sum}_o(q,s)$.

(113)[13] $\mathrm{diff}_o(a,b) = o$ iff $a = b$.

(114) $o, \mathrm{diff}_o(b,a) \upharpoonleft\!\upharpoonright a,b$.

(115) o, $\mathrm{diff}_o(b,a)$ and $\mathrm{diff}_o(d,c)$ are collinear iff $a,b \upharpoonleft\!\upharpoonright c,d$.

Let us consider S_1, a, b, c, d, o. We say that a, b, c, d form a trapezium with vertex o if and only if:

(Def. 8) o, a and c are not collinear and o, a and b are collinear and o, c and d are collinear and $a,c \upharpoonleft\!\upharpoonright b,d$.

Let us consider S_1, o, p. We say that there are trapeziums through p with vertex o if and only if:

(Def. 9) For all b, c there exists d such that if o, p and b are collinear, then o, c and d are collinear and $p,c \upharpoonleft\!\upharpoonright b,d$.

One can prove the following propositions:

(118)[14] If a, b, c, d form a trapezium with vertex o, then $o \neq a$ and $a \neq c$ and $c \neq o$.

(119) Suppose a, b, c, x form a trapezium with vertex o and a, b, c, y form a trapezium with vertex o. Then $x = y$.

(120) If o, a and b are not collinear, then a, o, b, o form a trapezium with vertex o.

(121) If a, b, c, d form a trapezium with vertex o, then c, d, a, b form a trapezium with vertex o.

(122) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then $o \neq d$.

(123) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then o, b and d are not collinear.

[11] The propositions (95)–(98) have been removed.
[12] The proposition (105) has been removed.
[13] The proposition (112) has been removed.
[14] The propositions (116) and (117) have been removed.

(124) Suppose $o \neq b$ and a, b, c, d form a trapezium with vertex o. Then b, a, d, c form a trapezium with vertex o.

(125) If $o = b$ or $o = d$ and if a, b, c, d form a trapezium with vertex o, then $o = b$ and $o = d$.

(126) Suppose a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o. Then $b, c \mathrel{\upharpoonleft\!\upharpoonright} q, r$.

(127) Suppose that

(i) a, p, b, q form a trapezium with vertex o,

(ii) a, p, c, r form a trapezium with vertex o, and

(iii) o, b and c are not collinear.

Then b, q, c, r form a trapezium with vertex o.

(128) Suppose that

(i) a, p, b, q form a trapezium with vertex o,

(ii) a, p, c, r form a trapezium with vertex o, and

(iii) b, q, d, s form a trapezium with vertex o.

Then $c, d \mathrel{\upharpoonleft\!\upharpoonright} r, s$.

(129) For all o, a there exists p such that o, a and p are collinear and there are trapeziums through p with vertex o.

(130) There exist x, y, z such that $x \neq y$ and $y \neq z$ and $z \neq x$.

(131) If there are trapeziums through p with vertex o, then $o \neq p$.

(132) Suppose there are trapeziums through p with vertex o. Then there exists q such that o, p and q are not collinear and there are trapeziums through q with vertex o.

(133) Suppose that

(i) o, p and c are not collinear,

(ii) o, p and b are collinear, and

(iii) there are trapeziums through p with vertex o.

Then there exists d such that p, b, c, d form a trapezium with vertex o.

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