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Linear Combinations in Real Unitary Space

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Summary. In this article, we mainly discuss linear combination of vectors in Real Unitary Space and dimension of the space. As the result, we obtain some theorems that are similar to those in Real Linear Space.

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The articles [10], [5], [17], [12], [2], [18], [1], [3], [4], [14], [11], [6], [13], [16], [9], [15], [8], and [7] provide the notation and terminology for this paper.

1. Definition and Fundamental Properties of Linear Combination

Let V be a real unitary space and let A be a subset of V. The functor $\mathrm{Lin}(A)$ yielding a strict subspace of V is defined as follows:

(Def. 1) The carrier of $\mathrm{Lin}(A) = \{\sum l : l$ ranges over linear combinations of $A\}$.

We now state a number of propositions:

(1) Let V be a real unitary space, A be a subset of V, and x be a set. Then $x \in \mathrm{Lin}(A)$ if and only if there exists a linear combination l of A such that $x = \sum l$.

(2) For every real unitary space V and for every subset A of V and for every set x such that $x \in A$ holds $x \in \mathrm{Lin}(A)$.

(3) For every real unitary space V holds $\mathrm{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(4) For every real unitary space V and for every subset A of V such that $\mathrm{Lin}(A) = \mathbf{0}_V$ holds $A = \emptyset$ or $A = \{0_V\}$.

(5) Let V be a real unitary space, W be a strict subspace of V, and A be a subset of V. If $A =$ the carrier of W, then $\mathrm{Lin}(A) = W$.

(6) For every strict real unitary space V and for every subset A of V such that $A =$ the carrier of V holds $\mathrm{Lin}(A) = V$.

(7) For every real unitary space V and for all subsets A, B of V such that $A \subseteq B$ holds $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(8) For every strict real unitary space V and for all subsets A, B of V such that $\mathrm{Lin}(A) = V$ and $A \subseteq B$ holds $\mathrm{Lin}(B) = V$.

(9) For every real unitary space V and for all subsets A, B of V holds $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(10) For every real unitary space V and for all subsets A, B of V holds $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(11) Let V be a real unitary space and A be a subset of V. Suppose A is linearly independent. Then there exists a subset B of V such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) =$ the unitary space structure of V.

(12) Let V be a real unitary space and A be a subset of V. Suppose $\mathrm{Lin}(A) = V$. Then there exists a subset B of V such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

2. Definition of the Basis of Real Unitary Space

Let V be a real unitary space. A subset of V is called a basis of V if:

(Def. 2) It is linearly independent and $\mathrm{Lin}(\mathrm{it}) =$ the unitary space structure of V.

We now state three propositions:

(13) Let V be a strict real unitary space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(14) For every real unitary space V and for every subset A of V such that $\mathrm{Lin}(A) = V$ there exists a basis I of V such that $I \subseteq A$.

(15) Let V be a real unitary space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

3. Some Theorems of Lin, Sum, Carrier

One can prove the following propositions:

(16) Let V be a real unitary space, L be a linear combination of V, A be a subset of V, and F be a finite sequence of elements of the carrier of V. Suppose $\mathrm{rng}\, F \subseteq$ the carrier of $\mathrm{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\, F) = \sum K$.

(17) Let V be a real unitary space, L be a linear combination of V, and A be a subset of V. Suppose the support of $L \subseteq$ the carrier of $\mathrm{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(18) Let V be a real unitary space, W be a subspace of V, and L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Let K be a linear combination of W. Suppose $K = L \restriction$the carrier of W. Then the support of $L =$ the support of K and $\sum L = \sum K$.

(19) Let V be a real unitary space, W be a subspace of V, and K be a linear combination of W. Then there exists a linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.

(20) Let V be a real unitary space, W be a subspace of V, and L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.

(21) For every real unitary space V and for every basis I of V and for every vector v of V holds $v \in \mathrm{Lin}(I)$.

(22) Let V be a real unitary space, W be a subspace of V, and A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(23) Let V be a real unitary space, W be a subspace of V, and A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(24) Let V be a real unitary space, W be a subspace of V, and A be a basis of W. Then there exists a basis B of V such that $A \subseteq B$.

(25) Let V be a real unitary space and A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \mathrm{Lin}(B)$.

(26) Let V be a real unitary space, I be a basis of V, and A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly dependent.

(27) Let V be a real unitary space, W be a subspace of V, and A be a subset of V. If $A \subseteq$ the carrier of W, then $\mathrm{Lin}(A)$ is a subspace of W.

(28) Let V be a real unitary space, W be a subspace of V, A be a subset of V, and B be a subset of W. If $A = B$, then $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

4. Subspaces of Real Unitary Space Generated by One, Two, or Three Vectors

Next we state a number of propositions:

(29) Let V be a real unitary space, v be a vector of V, and x be a set. Then $x \in \mathrm{Lin}(\{v\})$ if and only if there exists a real number a such that $x = a \cdot v$.

(30) For every real unitary space V and for every vector v of V holds $v \in \mathrm{Lin}(\{v\})$.

(31) Let V be a real unitary space, v, w be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w\})$ if and only if there exists a real number a such that $x = v + a \cdot w$.

(32) Let V be a real unitary space, w_1, w_2 be vectors of V, and x be a set. Then $x \in \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist real numbers a, b such that $x = a \cdot w_1 + b \cdot w_2$.

(33) For every real unitary space V and for all vectors w_1, w_2 of V holds $w_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2\})$.

(34) Let V be a real unitary space, v, w_1, w_2 be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist real numbers a, b such that $x = v + a \cdot w_1 + b \cdot w_2$.

(35) Let V be a real unitary space, v_1, v_2, v_3 be vectors of V, and x be a set. Then $x \in \mathrm{Lin}(\{v_1, v_2, v_3\})$ if and only if there exist real numbers a, b, c such that $x = a \cdot v_1 + b \cdot v_2 + c \cdot v_3$.

(36) For every real unitary space V and for all vectors w_1, w_2, w_3 of V holds $w_1 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_3 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$.

(37) Let V be a real unitary space, v, w_1, w_2, w_3 be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w_1, w_2, w_3\})$ if and only if there exist real numbers a, b, c such that $x = v + a \cdot w_1 + b \cdot w_2 + c \cdot w_3$.

(38) For every real unitary space V and for all vectors v, w of V such that $v \in \mathrm{Lin}(\{w\})$ and $v \neq 0_V$ holds $\mathrm{Lin}(\{v\}) = \mathrm{Lin}(\{w\})$.

(39) Let V be a real unitary space and v_1, v_2, w_1, w_2 be vectors of V. Suppose $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent and $v_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $v_2 \in \mathrm{Lin}(\{w_1, w_2\})$. Then $\mathrm{Lin}(\{w_1, w_2\}) = \mathrm{Lin}(\{v_1, v_2\})$ and $\{w_1, w_2\}$ is linearly independent and $w_1 \neq w_2$.

5. Auxiliary Theorems

We now state several propositions:

(40) For every real unitary space V and for every set x holds $x \in \mathbf{0}_V$ iff $x = 0_V$.

(41) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(42) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. Suppose W_1 is a subspace of W_2 and a subspace of W_3. Then W_1 is a subspace of $W_2 \cap W_3$.

(43) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. Suppose W_1 is a subspace of W_3 and W_2 is a subspace of W_3. Then $W_1 + W_2$ is a subspace of W_3.

(44) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(45) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v be a vector of V. If W_1 is a subspace of W_2, then $v + W_1 \subseteq v + W_2$.

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