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Operations on Subspaces in Real Unitary Space

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Summary. In this article, we extend an operation of real linear space to real unitary space. We show theorems proved in [8] on real unitary space.

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The articles [7], [3], [10], [11], [2], [1], [13], [12], [6], [9], [5], and [4] provide the notation and terminology for this paper.

1. Definitions of Sum and Intersection of Subspaces

Let V be a real unitary space and let W_1, W_2 be subspaces of V. The functor W_1+W_2 yields a strict subspace of V and is defined as follows:

(Def. 1) The carrier of $W_1+W_2=\{v+u; v$ ranges over vectors of V, u ranges over vectors of V: $v\in W_1 \wedge u\in W_2\}$.

Let V be a real unitary space and let W_1, W_2 be subspaces of V. The functor $W_1\cap W_2$ yielding a strict subspace of V is defined as follows:

(Def. 2) The carrier of $W_1\cap W_2=$ (the carrier of W_1) $\cap$ (the carrier of W_2).

2. Theorems of Sum and Intersecton of Subspaces

The following propositions are true:

(1) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. Then $x\in W_1+W_2$ if and only if there exist vectors v_1, v_2 of V such that $v_1\in W_1$ and $v_2\in W_2$ and $x=v_1+v_2$.

(2) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v be a vector of V. If $v\in W_1$ or $v\in W_2$, then $v\in W_1+W_2$.

(3) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. Then $x\in W_1\cap W_2$ if and only if $x\in W_1$ and $x\in W_2$.

(4) For every real unitary space V and for every strict subspace W of V holds $W+W=W$.

(5) For every real unitary space V and for all subspaces W_1, W_2 of V holds $W_1+W_2=W_2+W_1$.

(6) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(7) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(8) Let V be a real unitary space, W_1 be a subspace of V, and W_2 be a strict subspace of V. Then W_1 is a subspace of W_2 if and only if $W_1 + W_2 = W_2$.

(9) For every real unitary space V and for every strict subspace W of V holds $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(10) Let V be a real unitary space. Then $\mathbf{0}_V + \Omega_V =$ the unitary space structure of V and $\Omega_V + \mathbf{0}_V =$ the unitary space structure of V.

(11) Let V be a real unitary space and W be a subspace of V. Then $\Omega_V + W =$ the unitary space structure of V and $W + \Omega_V =$ the unitary space structure of V.

(12) For every strict real unitary space V holds $\Omega_V + \Omega_V = V$.

(13) For every real unitary space V and for every strict subspace W of V holds $W \cap W = W$.

(14) For every real unitary space V and for all subspaces W_1, W_2 of V holds $W_1 \cap W_2 = W_2 \cap W_1$.

(15) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(16) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(17) Let V be a real unitary space, W_2 be a subspace of V, and W_1 be a strict subspace of V. Then W_1 is a subspace of W_2 if and only if $W_1 \cap W_2 = W_1$.

(18) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(19) For every real unitary space V holds $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(20) For every real unitary space V and for every strict subspace W of V holds $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(21) For every strict real unitary space V holds $\Omega_V \cap \Omega_V = V$.

(22) For every real unitary space V and for all subspaces W_1, W_2 of V holds $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(23) For every real unitary space V and for every subspace W_1 of V and for every strict subspace W_2 of V holds $W_1 \cap W_2 + W_2 = W_2$.

(24) For every real unitary space V and for every subspace W_1 of V and for every strict subspace W_2 of V holds $W_2 \cap (W_2 + W_1) = W_2$.

(25) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(26) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(27) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(28) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(29) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a strict subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(30) For every real unitary space V and for all strict subspaces W_1, W_2 of V holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(31) Let V be a real unitary space, W_1 be a subspace of V, and W_2, W_3 be strict subspaces of V. If W_1 is a subspace of W_2, then $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(32) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then there exists a subspace W of V such that the carrier of $W =$ (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

3. Introduction of a Set of Subspaces of Real Unitary Space

Let V be a real unitary space. The functor Subspaces V yielding a set is defined as follows:

(Def. 3) For every set x holds $x \in$ Subspaces V iff x is a strict subspace of V.

Let V be a real unitary space. Note that Subspaces V is non empty.
The following proposition is true

(33) For every strict real unitary space V holds $V \in$ Subspaces V.

4. Definition of the Direct Sum and Linear Complement of Subspaces

Let V be a real unitary space and let W_1, W_2 be subspaces of V. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def. 4) The unitary space structure of $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let V be a real unitary space and let W be a subspace of V. A subspace of V is called a linear complement of W if:

(Def. 5) V is the direct sum of it and W.

Let V be a real unitary space and let W be a subspace of V. Observe that there exists a linear complement of W which is strict.
The following two propositions are true:

(34) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Then W_2 is a linear complement of W_1.

(35) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then V is the direct sum of L and W and the direct sum of W and L.

5. Theorems Concerning the Sum, Linear Complement and Coset of Subspace

One can prove the following propositions:

(36) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then $W + L =$ the unitary space structure of V and $L + W =$ the unitary space structure of V.

(37) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

(38) Let V be a real unitary space and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(39) Every real unitary space V is the direct sum of $\mathbf{0}_V$ and Ω_V and the direct sum of Ω_V and $\mathbf{0}_V$.

(40) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then W is a linear complement of L.

(41) For every real unitary space V holds $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

(42) Let V be a real unitary space, W_1, W_2 be subspaces of V, C_1 be a coset of W_1, and C_2 be a coset of W_2. If C_1 meets C_2, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(43) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then V is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists a vector v of V such that $C_1 \cap C_2 = \{v\}$.

6. Decomposition of a Vector of Real Unitary Space

Next we state three propositions:

(44) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then $W_1 + W_2 =$ the unitary space structure of V if and only if for every vector v of V there exist vectors v_1, v_2 of V such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(45) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v, v_1, v_2, u_1, u_2 be vectors of V. Suppose V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$. Then $v_1 = u_1$ and $v_2 = u_2$.

(46) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose that

(i) $V = W_1 + W_2$, and

(ii) there exists a vector v of V such that for all vectors v_1, v_2, u_1, u_2 of V such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$.

Then V is the direct sum of W_1 and W_2.

Let V be a real unitary space, let v be a vector of V, and let W_1, W_2 be subspaces of V. Let us assume that V is the direct sum of W_1 and W_2. The functor $v_{\langle W_1, W_2 \rangle}$ yields an element of $[:$the carrier of V, the carrier of $V:]$ and is defined as follows:

(Def. 6) $v = (v_{\langle W_1,W_2 \rangle})_{\mathbf{1}} + (v_{\langle W_1,W_2 \rangle})_{\mathbf{2}}$ and $(v_{\langle W_1,W_2 \rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1,W_2 \rangle})_{\mathbf{2}} \in W_2$.

One can prove the following propositions:

(47) Let V be a real unitary space, v be a vector of V, and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2 \rangle})_{\mathbf{1}} = (v_{\langle W_2,W_1 \rangle})_{\mathbf{2}}$.

(48) Let V be a real unitary space, v be a vector of V, and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2 \rangle})_{\mathbf{2}} = (v_{\langle W_2,W_1 \rangle})_{\mathbf{1}}$.

(49) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, v be a vector of V, and t be an element of $[:$the carrier of V, the carrier of $V:]$. If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v_{\langle W,L \rangle}$.

(50) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} + (v_{\langle W,L \rangle})_{\mathbf{2}} = v$.

(51) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} \in W$ and $(v_{\langle W,L \rangle})_{\mathbf{2}} \in L$.

(52) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} = (v_{\langle L,W \rangle})_{\mathbf{2}}$.

(53) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{2}} = (v_{\langle L,W \rangle})_{\mathbf{1}}$.

7. Introduction of Operations on Set of Subspaces

Let V be a real unitary space. The functor $\mathrm{SubJoin}\,V$ yielding a binary operation on $\mathrm{Subspaces}\,V$ is defined as follows:

(Def. 7) For all elements A_1, A_2 of $\mathrm{Subspaces}\,V$ and for all subspaces W_1, W_2 of V such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\,V)(A_1, A_2) = W_1 + W_2$.

Let V be a real unitary space. The functor $\mathrm{SubMeet}\,V$ yielding a binary operation on $\mathrm{Subspaces}\,V$ is defined by:

(Def. 8) For all elements A_1, A_2 of $\mathrm{Subspaces}\,V$ and for all subspaces W_1, W_2 of V such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\,V)(A_1, A_2) = W_1 \cap W_2$.

8. Theorems of Functions SubJoin, SubMeet

One can prove the following proposition

(54) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a lattice.

Let V be a real unitary space. One can check that $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lattice-like.

The following propositions are true:

(55) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lower-bounded.

(56) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is upper-bounded.

(57) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a bound lattice.

(58) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is modular.

(59) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is complemented.

Let V be a real unitary space. Observe that $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lower-bounded, upper-bounded, modular, and complemented.

We now state the proposition

(60) Let V be a real unitary space and W_1, W_2, W_3 be strict subspaces of V. If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

9. Auxiliary Theorems in Real Unitary Space

One can prove the following propositions:

(61) Let V be a real unitary space and W be a strict subspace of V. Suppose that for every vector v of V holds $v \in W$. Then $W =$ the unitary space structure of V.

(62) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then there exists a coset C of W such that $v \in C$.

(63) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and x be a set. Then $x \in v + W$ if and only if there exists a vector u of V such that $u \in W$ and $x = v + u$.

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