JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Introduction to Probability

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Summary. Definitions of Elementary Event and Event in any sample space E are given. Next, the probability of an Event when E is finite is introduced and some properties of this function are investigated. Last part of the paper is devoted to the conditional probability and essential properties of this function (Bayes Theorem).

MML Identifier: RPR_1.

WWW: http://mizar.org/JFM/Vol2/rpr_1.html

The articles [8], [9], [4], [7], [6], [2], [5], [1], and [3] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: E denotes a non empty set, a denotes an element of E, A, B denote subsets of E, Y denotes a set, and p denotes a finite sequence.

Let E be a non empty set. Observe that there exists a subset of E which is non empty and trivial.

Let us consider E. An elementary event of E is a non empty trivial subset of E.

Next we state the proposition

(1) Let e be a non empty subset of E. Then e is an elementary event of E if and only if for every Y holds $Y \subseteq e$ iff $Y = \emptyset$ or $Y = e$.

Let us consider E. One can check that every elementary event of E is finite.

In the sequel e, e_1, e_2 are elementary events of E.

Next we state several propositions:

(5)[1] If $e = A \cup B$ and $A \neq B$, then $A = \emptyset$ and $B = e$ or $A = e$ and $B = \emptyset$.

(6) If $e = A \cup B$, then $A = e$ and $B = e$ or $A = e$ and $B = \emptyset$ or $A = \emptyset$ and $B = e$.

(7) $\{a\}$ is an elementary event of E.

(10)[2] If $e_1 \subseteq e_2$, then $e_1 = e_2$.

(11) There exists a such that $a \in E$ and $e = \{a\}$.

(12) There exists e which is an elementary event of E.

(14)[3] There exists p such that p is a finite sequence of elements of E and $\operatorname{rng} p = e$ and $\operatorname{len} p = 1$.

Let E be a set. An event of E is a subset of E.

Next we state several propositions:

[1] The propositions (2)–(4) have been removed.
[2] The propositions (8) and (9) have been removed.
[3] The proposition (13) has been removed.

(22)[4] Let E be a non empty set, e be an elementary event of E, and A be an event of E. Then e misses A or $e \cap A = e$.

(25)[5] For every non empty set E and for every event A of E such that $A \neq \emptyset$ there exists an elementary event e of E such that $e \subseteq A$.

(26) Let E be a non empty set, e be an elementary event of E, and A be an event of E. If $e \subseteq A \cup A^{c}$, then $e \subseteq A$ or $e \subseteq A^{c}$.

(27) $e_1 = e_2$ or e_1 misses e_2.

(34)[6] $A \cap B$ misses $A \cap B^{c}$.

Let E be a finite non empty set and let A be an event of E. The functor $P(A)$ yields a real number and is defined as follows:

(Def. 4)[7] $P(A) = \frac{\operatorname{card} A}{\operatorname{card} E}$.

We now state a number of propositions:

(38)[8] For every finite non empty set E and for every elementary event e of E holds $P(e) = \frac{1}{\operatorname{card} E}$.

(39) For every finite non empty set E holds $P(\Omega_E) = 1$.

(40) For every finite non empty set E holds $P(\emptyset_E) = 0$.

(41) For every finite non empty set E and for all events A, B of E such that A misses B holds $P(A \cap B) = 0$.

(42) For every finite non empty set E and for every event A of E holds $P(A) \leq 1$.

(43) For every finite non empty set E and for every event A of E holds $0 \leq P(A)$.

(44) For every finite non empty set E and for all events A, B of E such that $A \subseteq B$ holds $P(A) \leq P(B)$.

(46)[9] For every finite non empty set E and for all events A, B of E holds $P(A \cup B) = (P(A) + P(B)) - P(A \cap B)$.

(47) For every finite non empty set E and for all events A, B of E such that A misses B holds $P(A \cup B) = P(A) + P(B)$.

(48) For every finite non empty set E and for every event A of E holds $P(A) = 1 - P(A^{c})$ and $P(A^{c}) = 1 - P(A)$.

(49) For every finite non empty set E and for all events A, B of E holds $P(A \setminus B) = P(A) - P(A \cap B)$.

(50) For every finite non empty set E and for all events A, B of E such that $B \subseteq A$ holds $P(A \setminus B) = P(A) - P(B)$.

(51) For every finite non empty set E and for all events A, B of E holds $P(A \cup B) \leq P(A) + P(B)$.

(53)[10] For every finite non empty set E and for all events A, B of E holds $P(A) = P(A \cap B) + P(A \cap B^{c})$.

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[4] The propositions (15)–(21) have been removed.
[5] The propositions (23) and (24) have been removed.
[6] The propositions (28)–(33) have been removed.
[7] The definitions (Def. 1)–(Def. 3) have been removed.
[8] The propositions (35)–(37) have been removed.
[9] The proposition (45) has been removed.
[10] The proposition (52) has been removed.

(54) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) = \mathrm{P}(A \cup B) - \mathrm{P}(B \setminus A)$.

(55) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) + \mathrm{P}(A^{\mathrm{c}} \cap B) = \mathrm{P}(B) + \mathrm{P}(B^{\mathrm{c}} \cap A)$.

(56) For every finite non empty set E and for all events A, B, C of E holds $\mathrm{P}(A \cup B \cup C) = ((\mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)) - (\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C))) + \mathrm{P}(A \cap B \cap C)$.

(57) Let E be a finite non empty set and A, B, C be events of E. Suppose A misses B and A misses C and B misses C. Then $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)$.

(58) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) - \mathrm{P}(B) \leq \mathrm{P}(A \setminus B)$.

Let E be a finite non empty set and let B, A be events of E. The functor $\mathrm{P}(A/B)$ yielding a real number is defined by:

(Def. 5) $\mathrm{P}(A/B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

The following propositions are true:

(60)[11] For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A \cap B) = \mathrm{P}(A/B) \cdot \mathrm{P}(B)$.

(61) For every finite non empty set E and for every event A of E holds $\mathrm{P}(A/\Omega_E) = \mathrm{P}(A)$.

(62) For every finite non empty set E holds $\mathrm{P}(\Omega_E/\Omega_E) = 1$.

(63) For every finite non empty set E holds $\mathrm{P}(\emptyset_E/\Omega_E) = 0$.

(64) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A/B) \leq 1$.

(65) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $0 \leq \mathrm{P}(A/B)$.

(66) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A/B) = 1 - \frac{\mathrm{P}(B \setminus A)}{\mathrm{P}(B)}$.

(67) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ and $A \subseteq B$ holds $\mathrm{P}(A/B) = \frac{\mathrm{P}(A)}{\mathrm{P}(B)}$.

(68) For every finite non empty set E and for all events A, B of E such that A misses B holds $\mathrm{P}(A/B) = 0$.

(69) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(A)$ and $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A) \cdot \mathrm{P}(B/A) = \mathrm{P}(B) \cdot \mathrm{P}(A/B)$.

(70) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A/B) = 1 - \mathrm{P}(A^{\mathrm{c}}/B)$ and $\mathrm{P}(A^{\mathrm{c}}/B) = 1 - \mathrm{P}(A/B)$.

(71) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ and $B \subseteq A$ holds $\mathrm{P}(A/B) = 1$.

(72) For every finite non empty set E and for every event B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(\Omega_E/B) = 1$.

(73) For every finite non empty set E and for every event A of E such that $0 < \mathrm{P}(A)$ holds $\mathrm{P}(A^{\mathrm{c}}/A) = 0$.

[11] The proposition (59) has been removed.

(74) For every finite non empty set E and for every event A of E such that $P(A) < 1$ holds $P(A/A^c) = 0$.

(75) For every finite non empty set E and for all events A, B of E such that $0 < P(B)$ and A misses B holds $P(A^c/B) = 1$.

(76) Let E be a finite non empty set and A, B be events of E. If $0 < P(A)$ and $P(B) < 1$ and A misses B, then $P(A/B^c) = \frac{P(A)}{1-P(B)}$.

(77) Let E be a finite non empty set and A, B be events of E. If $0 < P(A)$ and $P(B) < 1$ and A misses B, then $P(A^c/B^c) = 1 - \frac{P(A)}{1-P(B)}$.

(78) For every finite non empty set E and for all events A, B, C of E such that $0 < P(B \cap C)$ and $0 < P(C)$ holds $P(A \cap B \cap C) = P(A/(B \cap C)) \cdot P(B/C) \cdot P(C)$.

(79) For every finite non empty set E and for all events A, B of E such that $0 < P(B)$ and $P(B) < 1$ holds $P(A) = P(A/B) \cdot P(B) + P(A/B^c) \cdot P(B^c)$.

(80) Let E be a finite non empty set and A, B_1, B_2 be events of E. If $0 < P(B_1)$ and $0 < P(B_2)$ and $B_1 \cup B_2 = E$ and B_1 misses B_2, then $P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2)$.

(81) Let E be a finite non empty set and A, B_1, B_2, B_3 be events of E. Suppose $0 < P(B_1)$ and $0 < P(B_2)$ and $0 < P(B_3)$ and $B_1 \cup B_2 \cup B_3 = E$ and B_1 misses B_2 and B_1 misses B_3 and B_2 misses B_3. Then $P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + P(A/B_3) \cdot P(B_3)$.

(82) Let E be a finite non empty set and A, B_1, B_2 be events of E. Suppose $0 < P(B_1)$ and $0 < P(B_2)$ and $B_1 \cup B_2 = E$ and B_1 misses B_2. Then $P(B_1/A) = \frac{P(A/B_1) \cdot P(B_1)}{P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2)}$.

(83) Let E be a finite non empty set and A, B_1, B_2, B_3 be events of E. Suppose $0 < P(B_1)$ and $0 < P(B_2)$ and $0 < P(B_3)$ and $B_1 \cup B_2 \cup B_3 = E$ and B_1 misses B_2 and B_1 misses B_3 and B_2 misses B_3. Then $P(B_1/A) = \frac{P(A/B_1) \cdot P(B_1)}{P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + P(A/B_3) \cdot P(B_3)}$.

Let E be a finite non empty set and let A, B be events of E. We say that A and B are independent if and only if:

(Def. 6) $P(A \cap B) = P(A) \cdot P(B)$.

Let us note that the predicate A and B are independent is symmetric.

Next we state four propositions:

(86)[12] Let E be a finite non empty set and A, B be events of E. If $0 < P(B)$ and A and B are independent, then $P(A/B) = P(A)$.

(87) For every finite non empty set E and for all events A, B of E such that $P(B) = 0$ holds A and B are independent.

(88) Let E be a finite non empty set and A, B be events of E. If A and B are independent, then A^c and B are independent.

(89) Let E be a finite non empty set and A, B be events of E. If A misses B and A and B are independent, then $P(A) = 0$ or $P(B) = 0$.

[12] The propositions (84) and (85) have been removed.

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Received June 13, 1990

Published January 2, 2004