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On the Two Short Axiomatizations of Ortholattices

Wioletta Truszkowska
University of Białystok

Adam Grabowski
University of Białystok

Summary. In the paper, two short axiom systems for Boolean algebras are introduced. In the first section we show that the single axiom (DN_1) proposed in [2] in terms of disjunction and negation characterizes Boolean algebras. To prove that (DN_1) is a single axiom for Robbins algebras (that is, Boolean algebras as well), we use the Otter theorem prover. The second section contains proof that the two classical axioms $(\mathrm{Meredith}_1)$, $(\mathrm{Meredith}_2)$ proposed by Meredith [3] may also serve as a basis for Boolean algebras. The results will be used to characterize ortholattices.

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The articles [4] and [1] provide the notation and terminology for this paper.

1. Single Axiom for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies (DN_1) if and only if:

(Def. 1) For all elements x, y, z, u of L holds $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x+(z^{\mathrm{c}}+(z+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=z$.

Let us mention that TrivComplLat satisfies (DN_1) and TrivOrtLat satisfies (DN_1).

One can verify that there exists a non empty complemented lattice structure which is join-commutative and join-associative and satisfies (DN_1).

Next we state a number of propositions:

(1) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u, v be elements of L. Then $((x+y)^{\mathrm{c}}+(((z+u)^{\mathrm{c}}+x)^{\mathrm{c}}+(y^{\mathrm{c}}+(y+v)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(2) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+(y^{\mathrm{c}}+(y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(3) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of L. Then $((x+x^{\mathrm{c}})^{\mathrm{c}}+x)^{\mathrm{c}}=x^{\mathrm{c}}$.

(4) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+(((y+y^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}}+(y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(5) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(6) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(7) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+(x+y)^{\mathrm{c}})^{\mathrm{c}}=x$.

(8) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+((x+y)^{\mathrm{c}}+x)^{\mathrm{c}})^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(9) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x+z)^{\mathrm{c}})^{\mathrm{c}}=z$.

(10) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=(y+x)^{\mathrm{c}}$.

(11) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}}=(x^{\mathrm{c}}+y)^{\mathrm{c}}$.

(12) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+((y+z)^{\mathrm{c}}+(z+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=(z+x)^{\mathrm{c}}$.

(13) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+(y^{\mathrm{c}}+(u+y)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(14) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}=(y+x)^{\mathrm{c}}$.

(15) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+(y+z)^{\mathrm{c}})^{\mathrm{c}}+z)^{\mathrm{c}}=(y+z)^{\mathrm{c}}$.

(16) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+((x+y)^{\mathrm{c}}+z)^{\mathrm{c}})^{\mathrm{c}}+z)^{\mathrm{c}}=((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}$.

(17) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y)^{\mathrm{c}}=(y+y)^{\mathrm{c}}$.

(18) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}}=x$.

(19) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+y^{\mathrm{c}})^{\mathrm{c}}=y$.

(20) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+(y+x^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=x^{\mathrm{c}}$.

(21) For every non empty complemented lattice structure L satisfying (DN_1) and for every element x of L holds $(x+x)^{\mathrm{c}}=x^{\mathrm{c}}$.

(22) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y)^{\mathrm{c}}=y^{\mathrm{c}}$.

(23) For every non empty complemented lattice structure L satisfying (DN_1) and for every element x of L holds $(x^{\mathrm{c}})^{\mathrm{c}}=x$.

(24) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y=(y^{\mathrm{c}})^{\mathrm{c}}$.

(25) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}})^{\mathrm{c}}=y+x$.

(26) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}}=((y+x)^{\mathrm{c}})^{\mathrm{c}}$.

(27) For every non empty complemented lattice structure L satisfying (DN_1) and for all elements x, y of L holds $x+y=y+x$.

Let us mention that every non empty complemented lattice structure which satisfies (DN_1) is also join-commutative.

The following propositions are true:

(28) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y=y$.

(29) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+y)^{\mathrm{c}}+x=x$.

(30) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $x+((y+x)^{\mathrm{c}}+y)^{\mathrm{c}}=x$.

(31) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y^{\mathrm{c}})^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}}=(x+y^{\mathrm{c}})^{\mathrm{c}}$.

(32) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}+(y+y^{\mathrm{c}})^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(33) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(34) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}}=(y^{\mathrm{c}}+y)^{\mathrm{c}}$.

(35) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y^{\mathrm{c}}+y)^{\mathrm{c}}=(y^{\mathrm{c}}+y)^{\mathrm{c}}$.

(36) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y^{\mathrm{c}}+z)^{\mathrm{c}}+y)^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(37) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}}=y+x$.

(38) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+((z+y)^{\mathrm{c}}+x)^{\mathrm{c}})^{\mathrm{c}}=(z+y)^{\mathrm{c}}+x$.

(39) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+((y+x)^{\mathrm{c}}+(y+z)^{\mathrm{c}})^{\mathrm{c}}=y+x$.

(40) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+y)^{\mathrm{c}}+((x+y)^{\mathrm{c}}+(x+z)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}+y=y$.

(41) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y^{\mathrm{c}}+z)^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(42) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+x^{\mathrm{c}}+z)^{\mathrm{c}}=x$.

(43) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x^{\mathrm{c}}+(y+x+z)^{\mathrm{c}}=x^{\mathrm{c}}$.

(44) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}+x=x+y^{\mathrm{c}}$.

(45) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+(x+y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(46) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+y)^{\mathrm{c}}+(x+z))^{\mathrm{c}}+y=y$.

(47) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y=((x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}$.

(48) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y=x^{\mathrm{c}}+y$.

(49) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}}+(((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z)=((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z)$.

(50) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}}+(y+x+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z)=((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z)$.

(51) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}}+(y+x+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z)=(y+x)+(y+z)$.

(52) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}})^{\mathrm{c}}+(y+z)=(y+x)+(y+z)$.

(53) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+y)+(x+z)=y+(x+z)$.

(54) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+y)+(x+z)=z+(x+y)$.

(55) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+z)=z+(y+x)$.

(56) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+z)=y+(z+x)$.

(57) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+y)+z=x+(y+z)$.

One can check that every non empty complemented lattice structure which satisfies (DN_1) is also join-associative and every non empty complemented lattice structure which satisfies (DN_1) is also Robbins.

One can prove the following two propositions:

(58) Let L be a non empty complemented lattice structure and x, z be elements of L. Suppose L is join-commutative, join-associative, and Huntington. Then $(z+x)*(z+x^{\mathrm{c}})=z$.

(59) Let L be a non empty complemented lattice structure such that L is join-commutative, join-associative, and Robbins. Then L satisfies (DN_1).

One can verify that every non empty complemented lattice structure which is join-commutative, join-associative, and Robbins satisfies also (DN_1).

One can verify that there exists a pre-ortholattice which is de Morgan and satisfies (DN_1).

One can check that every pre-ortholattice which is de Morgan satisfies (DN_1) is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also (DN_1).

2. Meredith Two Axioms for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies $(\mathrm{Meredith}_1)$ if and only if:

(Def. 2) For all elements x, y of L holds $(x^{\mathrm{c}}+y)^{\mathrm{c}}+x=x$.

We say that L satisfies $(\mathrm{Meredith}_2)$ if and only if:

(Def. 3) For all elements x, y, z of L holds $(x^{\mathrm{c}}+y)^{\mathrm{c}}+(z+y)=y+(z+x)$.

Let us observe that every non empty complemented lattice structure which satisfies $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$ is also join-commutative, join-associative, and Huntington and every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington satisfies also $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$.

Let us observe that there exists a pre-ortholattice which is de Morgan and satisfies $(\mathrm{Meredith}_1)$, $(\mathrm{Meredith}_2)$, and (DN_1).

One can check that every pre-ortholattice which is de Morgan satisfies $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$ is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$.

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