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Operations on Submodules in Right Module over Associative Ring

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The articles [7], [3], [9], [1], [10], [2], [11], [8], [6], [4], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: R denotes a ring, V denotes a right module over R, W, W_1, W_2, W_3 denote submodules of V, u, u_1, u_2, v, v_1, v_2 denote vectors of V, and x denotes a set.

Let us consider R, let us consider V, and let us consider W_1, W_2. The functor $W_1 + W_2$ yields a strict submodule of V and is defined as follows:

(Def. 1) The carrier of $W_1 + W_2 = \{v + u : v \in W_1 \ \wedge \ u \in W_2\}$.

Let us consider R, let us consider V, and let us consider W_1, W_2. The functor $W_1 \cap W_2$ yielding a strict submodule of V is defined as follows:

(Def. 2) The carrier of $W_1 \cap W_2 =$ (the carrier of W_1) $\cap$ (the carrier of W_2).

We now state a number of propositions:

(5)[1] $x \in W_1 + W_2$ iff there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ iff $x \in W_1$ and $x \in W_2$.

(8) For every strict submodule W of V holds $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a submodule of $W_1 + W_2$ and W_2 is a submodule of $W_1 + W_2$.

(12) For every strict submodule W_2 of V holds W_1 is a submodule of W_2 iff $W_1 + W_2 = W_2$.

(13) For every strict submodule W of V holds $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) For every strict right module V over R holds $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

[1] The propositions (1)–(4) have been removed.

(15) Let V be a right module over R and W be a submodule of V. Then $\Omega_V + W =$ the right module structure of V and $W + \Omega_V =$ the right module structure of V.

(16) For every strict right module V over R holds $\Omega_V + \Omega_V = V$.

(17) For every strict submodule W of V holds $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a submodule of W_1 and $W_1 \cap W_2$ is a submodule of W_2.

(21)(i) For every strict submodule W_1 of V such that W_1 is a submodule of W_2 holds $W_1 \cap W_2 = W_1$, and

(ii) for every W_1 such that $W_1 \cap W_2 = W_1$ holds W_1 is a submodule of W_2.

(22) If W_1 is a submodule of W_2, then $W_1 \cap W_3$ is a submodule of $W_2 \cap W_3$.

(23) If W_1 is a submodule of W_3, then $W_1 \cap W_2$ is a submodule of W_3.

(24) If W_1 is a submodule of W_2 and a submodule of W_3, then W_1 is a submodule of $W_2 \cap W_3$.

(25) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(27)[2] For every strict submodule W of V holds $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(28) For every strict right module V over R holds $\Omega_V \cap \Omega_V = V$.

(29) $W_1 \cap W_2$ is a submodule of $W_1 + W_2$.

(30) For every strict submodule W_2 of V holds $W_1 \cap W_2 + W_2 = W_2$.

(31) For every strict submodule W_1 of V holds $W_1 \cap (W_1 + W_2) = W_1$.

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a submodule of $W_2 \cap (W_1 + W_3)$.

(33) If W_1 is a submodule of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a submodule of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If W_1 is a submodule of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) For every strict submodule W_1 of V such that W_1 is a submodule of W_3 holds $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) For all strict submodules W_1, W_2 of V holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(38) For all strict submodules W_2, W_3 of V such that W_1 is a submodule of W_2 holds $W_1 + W_3$ is a submodule of $W_2 + W_3$.

(39) If W_1 is a submodule of W_2, then W_1 is a submodule of $W_2 + W_3$.

(40) If W_1 is a submodule of W_3 and W_2 is a submodule of W_3, then $W_1 + W_2$ is a submodule of W_3.

(41) There exists W such that the carrier of $W =$ (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a submodule of W_2 or W_2 is a submodule of W_1.

Let us consider R and let us consider V. The functor $\mathrm{Sub}(V)$ yielding a set is defined by:

(Def. 3) For every x holds $x \in \mathrm{Sub}(V)$ iff there exists a strict submodule W of V such that $W = x$.

[2] The proposition (26) has been removed.

Let us consider R and let us consider V. Observe that $\mathrm{Sub}(V)$ is non empty.
We now state the proposition

(44)[3] For every strict right module V over R holds $V \in \mathrm{Sub}(V)$.

Let us consider R, let us consider V, and let us consider W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def. 4) The right module structure of $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

The following two propositions are true:

(46)[4] If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(47) Every strict right module V over R is the direct sum of $\mathbf{0}_V$ and Ω_V and the direct sum of Ω_V and $\mathbf{0}_V$.

In the sequel C_1 is a coset of W_1 and C_2 is a coset of W_2.
The following propositions are true:

(48) If C_1 meets C_2, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(49) Let V be a right module over R and W_1, W_2 be submodules of V. Then V is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists a vector v of V such that $C_1 \cap C_2 = \{v\}$.

(50) Let V be a strict right module over R and W_1, W_2 be submodules of V. Then $W_1 + W_2 = V$ if and only if for every vector v of V there exist vectors v_1, v_2 of V such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(51) Let V be a right module over R, W_1, W_2 be submodules of V, and v, v_1, v_2, u_1, u_2 be vectors of V. Suppose V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$. Then $v_1 = u_1$ and $v_2 = u_2$.

(52) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

Let us consider R, let V be a right module over R, let v be a vector of V, and let W_1, W_2 be submodules of V. Let us assume that V is the direct sum of W_1 and W_2. The functor $v_{\langle W_1, W_2 \rangle}$ yielding an element of [: the carrier of V, the carrier of V :] is defined as follows:

(Def. 5) $v = (v_{\langle W_1, W_2 \rangle})_{\mathbf{1}} + (v_{\langle W_1, W_2 \rangle})_{\mathbf{2}}$ and $(v_{\langle W_1, W_2 \rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1, W_2 \rangle})_{\mathbf{2}} \in W_2$.

Next we state two propositions:

(57)[5] If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1, W_2 \rangle})_{\mathbf{1}} = (v_{\langle W_2, W_1 \rangle})_{\mathbf{2}}$.

(58) If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1, W_2 \rangle})_{\mathbf{2}} = (v_{\langle W_2, W_1 \rangle})_{\mathbf{1}}$.

In the sequel A_1, A_2 are elements of $\mathrm{Sub}(V)$.
Let us consider R and let us consider V. The functor $\mathrm{SubJoin}\,V$ yields a binary operation on $\mathrm{Sub}(V)$ and is defined by:

(Def. 6) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\,V)(A_1, A_2) = W_1 + W_2$.

[3] The propositions (42) and (43) have been removed.
[4] The proposition (45) has been removed.
[5] The propositions (53)–(56) have been removed.

Let us consider R and let us consider V. The functor $\mathrm{SubMeet}V$ yielding a binary operation on $\mathrm{Sub}(V)$ is defined by:

(Def. 7) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}V)(A_1, A_2) = W_1 \cap W_2$.

We now state several propositions:

(63)[6] $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a lattice.

(64) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a lower bound lattice.

(65) For every right module V over R holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is an upper bound lattice.

(66) For every right module V over R holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a bound lattice.

(67) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a modular lattice.

REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_2.html.

[5] Michał Muzalewski and Wojciech Skaba. Submodules and cosets of submodules in right module over associative ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rmod_2.html.

[6] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/domain_1.html.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[8] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[9] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[10] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[11] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

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[6] The propositions (59)–(62) have been removed.