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Properties of Partial Functions from a Domain to the Set of Real Numbers

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Summary. The article consists of two parts. In the first one we consider notion of nonnegative and nonpositive part of a real numbers. In the second we consider partial function from a domain to the set of real numbers (or more general to a domain). We define a few new operations for these functions and show connections between finite sequences of real numbers and functions which domain is finite. We introduce *integrations* for finite domain real valued functions.

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The articles [22], [26], [2], [23], [27], [5], [3], [4], [1], [12], [18], [20], [21], [8], [24], [28], [6], [7], [13], [16], [25], [10], [9], [19], [15], [14], [11], and [17] provide the notation and terminology for this paper.

1. Nonnegative and Nonpositive Part of a Real Number

In this paper n denotes a natural number and r denotes a real number.

Let n, m be natural numbers. Then $\min(n,m)$ is a natural number.

Let r be a real number. The functor $\max_+(r)$ yields a real number and is defined as follows:

(Def. 1) $\max_+(r) = \max(r,0)$.

The functor $\max_-(r)$ yields a real number and is defined as follows:

(Def. 2) $\max_-(r) = \max(-r,0)$.

The following propositions are true:

(1) For every real number r holds $r = \max_+(r) - \max_-(r)$.

(2) For every real number r holds $|r| = \max_+(r) + \max_-(r)$.

(3) For every real number r holds $2 \cdot \max_+(r) = r + |r|$.

(4) For all real numbers r, s such that $0 \leq r$ holds $\max_+(r \cdot s) = r \cdot \max_+(s)$.

(5) For all real numbers r, s holds $\max_+(r+s) \leq \max_+(r) + \max_+(s)$.

(6) For every real number r holds $0 \leq \max_+(r)$ and $0 \leq \max_-(r)$.

(7) For all real numbers r_1, r_2, s_1, s_2 such that $r_1 \leq s_1$ and $r_2 \leq s_2$ holds $\max(r_1,r_2) \leq \max(s_1,s_2)$.

2. Properties of Real Function

The following propositions are true:

(8) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r, s be real numbers. If $r \neq 0$, then $F^{-1}(\{\frac{s}{r}\}) = (r\,F)^{-1}(\{s\})$.

(9) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $(0\,F)^{-1}(\{0\}) = \operatorname{dom} F$.

(10) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $0 < r$, then $|F|^{-1}(\{r\}) = F^{-1}(\{-r, r\})$.

(11) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $|F|^{-1}(\{0\}) = F^{-1}(\{0\})$.

(12) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $r < 0$, then $|F|^{-1}(\{r\}) = \emptyset$.

(13) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, G be a partial function from C to $\mathbb{R}$, and r be a real number. Suppose $r \neq 0$. Then F and G are fiberwise equipotent if and only if $r\,F$ and $r\,G$ are fiberwise equipotent.

(14) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Then F and G are fiberwise equipotent if and only if $-F$ and $-G$ are fiberwise equipotent.

(15) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Suppose F and G are fiberwise equipotent. Then $|F|$ and $|G|$ are fiberwise equipotent.

Let X, Y be sets. Set of partial functions from X to Y is defined by:

(Def. 3) Every element of it is a partial function from X to Y.

Let X, Y be sets. Note that there exists a set of partial functions from X to Y which is non empty.

Let X, Y be sets. A non empty set of partial functions from X to Y is a non empty set of partial functions from X to Y.

Let X, Y be sets. Then $X \dot\rightarrow Y$ is a set of partial functions from X to Y. Let P be a non empty set of partial functions from X to Y. We see that the element of P is a partial function from X to Y.

Let D, C be non empty sets, let X be a subset of D, and let c be an element of C. Then $X \longmapsto c$ is an element of $D \dot\rightarrow C$.

Let D be a non empty set and let F_1, F_2 be elements of $D \dot\rightarrow \mathbb{R}$. Then $F_1 + F_2$ is an element of $D \dot\rightarrow \mathbb{R}$. Then $F_1 - F_2$ is an element of $D \dot\rightarrow \mathbb{R}$. Then $F_1\,F_2$ is an element of $D \dot\rightarrow \mathbb{R}$. Then $\frac{F_1}{F_2}$ is an element of $D \dot\rightarrow \mathbb{R}$.

Let D be a non empty set and let F be an element of $D \dot\rightarrow \mathbb{R}$. Then $|F|$ is an element of $D \dot\rightarrow \mathbb{R}$. Then $-F$ is an element of $D \dot\rightarrow \mathbb{R}$. Then $\frac{1}{F}$ is an element of $D \dot\rightarrow \mathbb{R}$.

Let D be a non empty set, let F be an element of $D \dot\rightarrow \mathbb{R}$, and let r be a real number. Then $r\,F$ is an element of $D \dot\rightarrow \mathbb{R}$.

Let D be a non empty set. The functor $+_{D \dot\rightarrow \mathbb{R}}$ yielding a binary operation on $D \dot\rightarrow \mathbb{R}$ is defined as follows:

(Def. 4) For all elements F_1, F_2 of $D \dot\rightarrow \mathbb{R}$ holds $+_{D \dot\rightarrow \mathbb{R}}(F_1, F_2) = F_1 + F_2$.

One can prove the following propositions:

(16) For every non empty set D holds $+_{D \dot\rightarrow \mathbb{R}}$ is commutative.

(17) For every non empty set D holds $+_{D \dot\rightarrow \mathbb{R}}$ is associative.

(18) For every non empty set D holds $\Omega_D \longmapsto (0$ **qua** real number$)$ is a unity w.r.t. $+_{D \dot\rightarrow \mathbb{R}}$.

(19) For every non empty set D holds $\mathbf{1}_{+_{D \dot{\to} \mathbb{R}}} = \Omega_D \longmapsto (0$ **qua** real number$)$.

(20) For every non empty set D holds $+_{D \dot{\to} \mathbb{R}}$ has a unity.

Let D be a non empty set and let f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$. The functor $\sum f$ yields an element of $D \dot{\to} \mathbb{R}$ and is defined by:

(Def. 5) $\sum f = +_{D \dot{\to} \mathbb{R}} \circledast f$.

One can prove the following propositions:

(21) For every non empty set D holds $\sum(\varepsilon_{(D \dot{\to} \mathbb{R})}) = \Omega_D \longmapsto (0$ **qua** real number$)$.

(22) For every non empty set D and for every element G of $D \dot{\to} \mathbb{R}$ holds $\sum\langle G \rangle = G$.

(23) Let D be a non empty set, f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and G be an element of $D \dot{\to} \mathbb{R}$. Then $\sum(f \frown \langle G \rangle) = \sum f + G$.

(24) For every non empty set D and for all finite sequences f_1, f_2 of elements of $D \dot{\to} \mathbb{R}$ holds $\sum(f_1 \frown f_2) = \sum f_1 + \sum f_2$.

(25) Let D be a non empty set, f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and G be an element of $D \dot{\to} \mathbb{R}$. Then $\sum(\langle G \rangle \frown f) = G + \sum f$.

(26) For every non empty set D and for all elements G_1, G_2 of $D \dot{\to} \mathbb{R}$ holds $\sum\langle G_1, G_2 \rangle = G_1 + G_2$.

(27) For every non empty set D and for all elements G_1, G_2, G_3 of $D \dot{\to} \mathbb{R}$ holds $\sum\langle G_1, G_2, G_3 \rangle = G_1 + G_2 + G_3$.

(28) Let D be a non empty set and f, g be finite sequences of elements of $D \dot{\to} \mathbb{R}$. If f and g are fiberwise equipotent, then $\sum f = \sum g$.

Let D be a non empty set and let f be a finite sequence. The functor $\mathrm{CHI}(f, D)$ yields a finite sequence of elements of $D \dot{\to} \mathbb{R}$ and is defined as follows:

(Def. 6) $\mathrm{len}\,\mathrm{CHI}(f, D) = \mathrm{len}\, f$ and for every n such that $n \in \mathrm{dom}\,\mathrm{CHI}(f, D)$ holds $(\mathrm{CHI}(f, D))(n) = \chi_{f(n), D}$.

Let D be a non empty set, let f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and let R be a finite sequence of elements of $\mathbb{R}$. The functor $R\, f$ yielding a finite sequence of elements of $D \dot{\to} \mathbb{R}$ is defined by the conditions (Def. 7).

(Def. 7)(i) $\mathrm{len}(R\, f) = \min(\mathrm{len}\, R, \mathrm{len}\, f)$, and

(ii) for every n such that $n \in \mathrm{dom}(R\, f)$ and for every partial function F from D to $\mathbb{R}$ and for every r such that $r = R(n)$ and $F = f(n)$ holds $(R\, f)(n) = r\, F$.

Let D be a non empty set, let f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and let d be an element of D. The functor $f \# d$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by the conditions (Def. 8).

(Def. 8)(i) $\mathrm{len}(f \# d) = \mathrm{len}\, f$, and

(ii) for every natural number n and for every element G of $D \dot{\to} \mathbb{R}$ such that $n \in \mathrm{dom}(f \# d)$ and $f(n) = G$ holds $(f \# d)(n) = G(d)$.

Let D, C be non empty sets, let f be a finite sequence of elements of $D \dot{\to} C$, and let d be an element of D. We say that d is common for dom f if and only if:

(Def. 9) For every element G of $D \dot{\to} C$ and for every natural number n such that $n \in \mathrm{dom}\, f$ and $f(n) = G$ holds $d \in \mathrm{dom}\, G$.

The following propositions are true:

(29) Let D, C be non empty sets, f be a finite sequence of elements of $D \dot{\to} C$, d be an element of D, and n be a natural number. If d is common for dom f and $n \neq 0$, then d is common for dom $f \restriction n$.

(30) Let D, C be non empty sets, f be a finite sequence of elements of $D \dot{\to} C$, d be an element of D, and n be a natural number. If d is common for dom f, then d is common for dom $f_{\restriction n}$.

(31) Let D be a non empty set, d be an element of D, and f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$. If $\operatorname{len} f \neq 0$, then d is common for dom f iff $d \in \operatorname{dom} \sum f$.

(32) Let D be a non empty set, f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, d be an element of D, and n be a natural number. Then $(f \restriction n) \# d = (f \# d) \restriction n$.

(33) For every non empty set D and for every finite sequence f and for every element d of D holds d is common for dom $\mathrm{CHI}(f, D)$.

(34) Let D be a non empty set, d be an element of D, f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and R be a finite sequence of elements of $\mathbb{R}$. If d is common for dom f, then d is common for dom $R\, f$.

(35) Let D be a non empty set, f be a finite sequence, R be a finite sequence of elements of $\mathbb{R}$, and d be an element of D. Then d is common for dom $R\, \mathrm{CHI}(f, D)$.

(36) Let D be a non empty set, d be an element of D, and f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$. If d is common for dom f, then $(\sum f)(d) = \sum (f \# d)$.

Let D be a non empty set and let F be a partial function from D to $\mathbb{R}$. The functor $\max_+(F)$ yields a partial function from D to $\mathbb{R}$ and is defined by:

(Def. 10) $\operatorname{dom} \max_+(F) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom} \max_+(F)$ holds $(\max_+(F))(d) = \max_+(F(d))$.

The functor $\max_-(F)$ yielding a partial function from D to $\mathbb{R}$ is defined by:

(Def. 11) $\operatorname{dom} \max_-(F) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom} \max_-(F)$ holds $(\max_-(F))(d) = \max_-(F(d))$.

The following propositions are true:

(37) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F = \max_+(F) - \max_-(F)$ and $|F| = \max_+(F) + \max_-(F)$ and $2 \max_+(F) = F + |F|$.

(38) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $0 < r$, then $F^{-1}(\{r\}) = (\max_+(F))^{-1}(\{r\})$.

(39) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}(]-\infty, 0]) = (\max_+(F))^{-1}(\{0\})$.

(40) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d \in \operatorname{dom} F$, then $0 \leq (\max_+(F))(d)$.

(41) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $0 < r$, then $F^{-1}(\{-r\}) = (\max_-(F))^{-1}(\{r\})$.

(42) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}([0, +\infty[) = (\max_-(F))^{-1}(\{0\})$.

(43) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d \in \operatorname{dom} F$, then $0 \leq (\max_-(F))(d)$.

(44) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Suppose F and G are fiberwise equipotent. Then $\max_+(F)$ and $\max_+(G)$ are fiberwise equipotent.

(45) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Suppose F and G are fiberwise equipotent. Then $\max_-(F)$ and $\max_-(G)$ are fiberwise equipotent.

Let A, B be sets. Note that there exists a partial function from A to B which is finite.

Let D be a non empty set and let F be a finite partial function from D to $\mathbb{R}$. One can check that $\max_+(F)$ is finite and $\max_-(F)$ is finite.

Next we state several propositions:

(46) Let D, C be non empty sets, F be a finite partial function from D to $\mathbb{R}$, and G be a finite partial function from C to $\mathbb{R}$. Suppose $\max_+(F)$ and $\max_+(G)$ are fiberwise equipotent and $\max_-(F)$ and $\max_-(G)$ are fiberwise equipotent. Then F and G are fiberwise equipotent.

(47) For every non empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_+(F)\restriction X = \max_+(F\restriction X)$.

(48) For every non empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_-(F)\restriction X = \max_-(F\restriction X)$.

(49) Let D be a non empty set and F be a partial function from D to $\mathbb{R}$. If for every element d of D such that $d \in \mathrm{dom} F$ holds $F(d) \geq 0$, then $\max_+(F) = F$.

(50) Let D be a non empty set and F be a partial function from D to $\mathbb{R}$. If for every element d of D such that $d \in \mathrm{dom} F$ holds $F(d) \leq 0$, then $\max_-(F) = -F$.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let r be a real number. The functor $F - r$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def. 12) $\mathrm{dom}(F - r) = \mathrm{dom} F$ and for every element d of D such that $d \in \mathrm{dom}(F - r)$ holds $(F - r)(d) = F(d) - r$.

Next we state four propositions:

(51) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F - 0 = F$.

(52) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, r be a real number, and X be a set. Then $F\restriction X - r = (F - r)\restriction X$.

(53) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r, s be real numbers. Then $F^{-1}(\{s + r\}) = (F - r)^{-1}(\{s\})$.

(54) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, G be a partial function from C to $\mathbb{R}$, and r be a real number. Then F and G are fiberwise equipotent if and only if $F - r$ and $G - r$ are fiberwise equipotent.

Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let X be a set. We say that F is convex on X if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) $X \subseteq \mathrm{dom} F$, and

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1 - p) \cdot s \in X$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

The following propositions are true:

(55) Let a, b be real numbers and F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \mathrm{dom} F$, and

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in [a, b]$ and $s \in [a, b]$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

(56) Let a, b be real numbers and F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a,b]$ if and only if the following conditions are satisfied:

(i) $[a,b] \subseteq \operatorname{dom} F$, and

(ii) for all real numbers x_1, x_2, x_3 such that $x_1 \in [a,b]$ and $x_2 \in [a,b]$ and $x_3 \in [a,b]$ and $x_1 < x_2$ and $x_2 < x_3$ holds $\frac{F(x_1)-F(x_2)}{x_1-x_2} \leq \frac{F(x_2)-F(x_3)}{x_2-x_3}$.

(57) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and X, Y be sets. If F is convex on X and $Y \subseteq X$, then F is convex on Y.

(58) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, X be a set, and r be a real number. Then F is convex on X if and only if $F - r$ is convex on X.

(59) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, X be a set, and r be a real number. Suppose $0 < r$. Then F is convex on X if and only if $r\,F$ is convex on X.

(60) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that $X \subseteq \operatorname{dom} F$ holds $0\,F$ is convex on X.

(61) Let F, G be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and X be a set. Suppose F is convex on X and G is convex on X. Then $F + G$ is convex on X.

(62) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, X be a set, and r be a real number. If F is convex on X, then $\max_+(F - r)$ is convex on X.

(63) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and X be a set. If F is convex on X, then $\max_+(F)$ is convex on X.

(64) $\mathrm{id}_{\Omega_{\mathbb{R}}}$ is convex on $\mathbb{R}$.

(65) For every real number r holds $\max_+(\mathrm{id}_{\Omega_{\mathbb{R}}} - r)$ is convex on $\mathbb{R}$.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let X be a set. Let us assume that $\operatorname{dom}(F \restriction X)$ is finite. The functor $\mathrm{FinS}(F, X)$ yielding a non-increasing finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 14) $F \restriction X$ and $\mathrm{FinS}(F, X)$ are fiberwise equipotent.

We now state two propositions:

(66) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X be a set. If $\operatorname{dom}(F \restriction X)$ is finite, then $\mathrm{FinS}(F, \operatorname{dom}(F \restriction X)) = \mathrm{FinS}(F, X)$.

(67) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X be a set. If $\operatorname{dom}(F \restriction X)$ is finite, then $\mathrm{FinS}(F \restriction X, X) = \mathrm{FinS}(F, X)$.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let X be a finite set. Then $F \restriction X$ is a finite partial function from D to $\mathbb{R}$.

The following propositions are true:

(68) Let D be a non empty set, d be an element of D, X be a set, and F be a partial function from D to $\mathbb{R}$. Suppose X is finite and $d \in \operatorname{dom}(F \restriction X)$. Then $(\mathrm{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F \restriction X$ are fiberwise equipotent.

(69) Let D be a non empty set, d be an element of D, X be a set, and F be a partial function from D to $\mathbb{R}$. Suppose $\operatorname{dom}(F \restriction X)$ is finite and $d \in \operatorname{dom}(F \restriction X)$. Then $(\mathrm{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F \restriction X$ are fiberwise equipotent.

(70) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, and Y be a finite set. If $Y = \operatorname{dom}(F \restriction X)$, then $\operatorname{len} \mathrm{FinS}(F, X) = \operatorname{card} Y$.

(71) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $\mathrm{FinS}(F,\emptyset) = \varepsilon_{\mathbb{R}}$.

(72) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d \in \mathrm{dom}\,F$, then $\mathrm{FinS}(F,\{d\}) = \langle F(d) \rangle$.

(73) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, and d be an element of D. If $\mathrm{dom}(F \restriction X)$ is finite and $d \in \mathrm{dom}(F \restriction X)$ and $(\mathrm{FinS}(F,X))(\mathrm{len}\,\mathrm{FinS}(F,X)) = F(d)$, then $\mathrm{FinS}(F,X) = (\mathrm{FinS}(F,X \setminus \{d\})) \frown \langle F(d) \rangle$.

(74) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. Suppose $\mathrm{dom}(F \restriction X)$ is finite and $Y \subseteq X$ and for all elements d_1, d_2 of D such that $d_1 \in \mathrm{dom}(F \restriction Y)$ and $d_2 \in \mathrm{dom}(F \restriction (X \setminus Y))$ holds $F(d_1) \geq F(d_2)$. Then $\mathrm{FinS}(F,X) = (\mathrm{FinS}(F,Y)) \frown \mathrm{FinS}(F,X \setminus Y)$.

(75) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, r be a real number, X be a set, and d be an element of D. Suppose $\mathrm{dom}(F \restriction X)$ is finite and $d \in \mathrm{dom}(F \restriction X)$. Then $(\mathrm{FinS}(F - r,X))(\mathrm{len}\,\mathrm{FinS}(F - r,X)) = (F - r)(d)$ if and only if $(\mathrm{FinS}(F,X))(\mathrm{len}\,\mathrm{FinS}(F,X)) = F(d)$.

(76) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, r be a real number, X be a set, and Z be a finite set. If $Z = \mathrm{dom}(F \restriction X)$, then $\mathrm{FinS}(F - r,X) = \mathrm{FinS}(F,X) - \mathrm{card}\,Z \mapsto r$.

(77) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X be a set. Suppose $\mathrm{dom}(F \restriction X)$ is finite and for every element d of D such that $d \in \mathrm{dom}(F \restriction X)$ holds $F(d) \geq 0$. Then $\mathrm{FinS}(\max_{+}(F),X) = \mathrm{FinS}(F,X)$.

(78) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, r be a real number, and Z be a finite set. If $Z = \mathrm{dom}(F \restriction X)$ and $\mathrm{rng}(F \restriction X) = \{r\}$, then $\mathrm{FinS}(F,X) = \mathrm{card}\,Z \mapsto r$.

(79) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. Suppose $\mathrm{dom}(F \restriction (X \cup Y))$ is finite and X misses Y. Then $\mathrm{FinS}(F,X \cup Y)$ and $(\mathrm{FinS}(F,X)) \frown \mathrm{FinS}(F,Y)$ are fiberwise equipotent.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let X be a set. The functor $\sum_{\kappa=0}^{X} F(\kappa)$ yielding a real number is defined by:

(Def. 15) $\sum_{\kappa=0}^{X} F(\kappa) = \sum \mathrm{FinS}(F,X)$.

Next we state several propositions:

(80) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, and r be a real number. If $\mathrm{dom}(F \restriction X)$ is finite, then $\sum_{\kappa=0}^{X} (r\,F)(\kappa) = r \cdot \sum_{\kappa=0}^{X} F(\kappa)$.

(81) Let D be a non empty set, F, G be partial functions from D to $\mathbb{R}$, X be a set, and Y be a finite set. If $Y = \mathrm{dom}(F \restriction X)$ and $\mathrm{dom}(F \restriction X) = \mathrm{dom}(G \restriction X)$, then $\sum_{\kappa=0}^{X} (F + G)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{X} G(\kappa)$.

(82) Let D be a non empty set, F, G be partial functions from D to $\mathbb{R}$, and X be a set. If $\mathrm{dom}(F \restriction X)$ is finite and $\mathrm{dom}(F \restriction X) = \mathrm{dom}(G \restriction X)$, then $\sum_{\kappa=0}^{X} (F - G)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) - \sum_{\kappa=0}^{X} G(\kappa)$.

(83) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, r be a real number, and Y be a finite set. If $Y = \mathrm{dom}(F \restriction X)$, then $\sum_{\kappa=0}^{X} (F - r)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) - r \cdot \mathrm{card}\,Y$.

(84) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $\sum_{\kappa=0}^{\emptyset} F(\kappa) = 0$.

(85) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d \in \mathrm{dom}\,F$, then $\sum_{\kappa=0}^{\{d\}} F(\kappa) = F(d)$.

(86) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. If $\mathrm{dom}(F \restriction (X \cup Y))$ is finite and X misses Y, then $\sum_{\kappa=0}^{X \cup Y} F(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{Y} F(\kappa)$.

(87) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. If $\mathrm{dom}(F \restriction (X \cup Y))$ is finite and $\mathrm{dom}(F \restriction X)$ misses $\mathrm{dom}(F \restriction Y)$, then $\sum_{\kappa=0}^{X \cup Y} F(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{Y} F(\kappa)$.

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