JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Properties of Real Functions

Jarosław Kotowicz
Warsaw University
Białystok

Summary. The list of theorems concerning properties of real sequences and functions is enlarged. (See e.g. [8], [5], [10]). The monotone real functions are introduced and their properties are discussed.

MML Identifier: RFUNCT_2.

WWW: http://mizar.org/JFM/Vol2/rfunct_2.html

The articles [12], [14], [1], [13], [4], [15], [2], [8], [6], [5], [16], [3], [9], [11], [10], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x, X, X_1, Y are sets, g, r, r_1, r_2, p are elements of $\mathbb{R}$, R is a subset of $\mathbb{R}$, s_1, s_2, s_3, s_4 are sequences of real numbers, N_1 is an increasing sequence of naturals, n is an element of $\mathbb{N}$, and h, h_1, h_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(2)[1] For all functions F, G and for every X holds $(G{\restriction}F^{\circ}X)\cdot(F{\restriction}X) = (G\cdot F){\restriction}X$.

(3) For all functions F, G and for all X, X_1 holds $(G{\restriction}X_1)\cdot(F{\restriction}X) = (G\cdot F){\restriction}(X\cap F^{-1}(X_1))$.

(4) For all functions F, G and for every X holds $X \subseteq \mathrm{dom}(G\cdot F)$ iff $X \subseteq \mathrm{dom}\,F$ and $F^{\circ}X \subseteq \mathrm{dom}\,G$.

(5) For every function F and for every X holds $(F{\restriction}X)^{\circ}X = F^{\circ}X$.

Let us consider s_1. Then $\mathrm{rng}\,s_1$ is a subset of $\mathbb{R}$.

We now state a number of propositions:

(6) $s_2 = s_3 - s_4$ iff for every n holds $s_2(n) = s_3(n) - s_4(n)$.

(7) $\mathrm{rng}(s_1 \uparrow n) \subseteq \mathrm{rng}\,s_1$.

(8) If $\mathrm{rng}\,s_1 \subseteq \mathrm{dom}\,h$, then $s_1(n) \in \mathrm{dom}\,h$.

(9) $x \in \mathrm{rng}\,s_1$ iff there exists n such that $x = s_1(n)$.

(10) $s_1(n) \in \mathrm{rng}\,s_1$.

(11) If s_2 is a subsequence of s_1, then $\mathrm{rng}\,s_2 \subseteq \mathrm{rng}\,s_1$.

(12) If s_2 is a subsequence of s_1 and s_1 is non-zero, then s_2 is non-zero.

(13) $(s_2+s_3)\cdot N_1 = s_2\cdot N_1 + s_3\cdot N_1$ and $(s_2-s_3)\cdot N_1 = s_2\cdot N_1 - s_3\cdot N_1$ and $(s_2\,s_3)\cdot N_1 = (s_2\cdot N_1)\,(s_3\cdot N_1)$.

[1] The proposition (1) has been removed.

(14) $(p\,s_1)\cdot N_1 = p\,(s_1\cdot N_1)$.

(15) $(-s_1)\cdot N_1 = -s_1\cdot N_1$ and $|s_1|\cdot N_1 = |s_1\cdot N_1|$.

(16) $(s_1\cdot N_1)^{-1} = {s_1}^{-1}\cdot N_1$.

(17) $(s_2/s_1)\cdot N_1 = (s_2\cdot N_1)/(s_1\cdot N_1)$.

(18) If s_1 is convergent and for every n holds $s_1(n) \leq 0$, then $\lim s_1 \leq 0$.

(19) If for every n holds $s_1(n) \in Y$, then $\operatorname{rng} s_1 \subseteq Y$.

Let us consider h, s_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$. The functor $h\cdot s_1$ yielding a sequence of real numbers is defined as follows:

(Def. 1) $h\cdot s_1 = (h$ **qua** function$)\cdot(s_1)$.

One can prove the following propositions:

(21)[2] If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(h\cdot s_1)(n) = h(s_1(n))$.

(22) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(h\cdot s_1)\uparrow n = h\cdot(s_1\uparrow n)$.

(23) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1+h_2)\cdot s_1 = h_1\cdot s_1 + h_2\cdot s_1$ and $(h_1-h_2)\cdot s_1 = h_1\cdot s_1 - h_2\cdot s_1$ and $(h_1\,h_2)\cdot s_1 = (h_1\cdot s_1)\,(h_2\cdot s_1)$.

(24) For every real number r such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$ holds $(r\,h)\cdot s_1 = r\,(h\cdot s_1)$.

(25) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $|h\cdot s_1| = |h|\cdot s_1$ and $-h\cdot s_1 = (-h)\cdot s_1$.

(26) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(\frac{1}{h})$, then $h\cdot s_1$ is non-zero.

(27) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(\frac{1}{h})$, then $\frac{1}{h}\cdot s_1 = (h\cdot s_1)^{-1}$.

(28) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(h\cdot s_1)\cdot N_1 = h\cdot(s_1\cdot N_1)$.

(29) If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$ and s_3 is a subsequence of s_2, then $h\cdot s_3$ is a subsequence of $h\cdot s_2$.

(30) If h is total, then $(h\cdot s_1)(n) = h(s_1(n))$.

(31) If h is total, then $h\cdot(s_1\uparrow n) = (h\cdot s_1)\uparrow n$.

(32) If h_1 is total and h_2 is total, then $(h_1+h_2)\cdot s_1 = h_1\cdot s_1 + h_2\cdot s_1$ and $(h_1-h_2)\cdot s_1 = h_1\cdot s_1 - h_2\cdot s_1$ and $(h_1\,h_2)\cdot s_1 = (h_1\cdot s_1)\,(h_2\cdot s_1)$.

(33) If h is total, then $(r\,h)\cdot s_1 = r\,(h\cdot s_1)$.

(34) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$, then $(h\restriction X)\cdot s_1 = h\cdot s_1$.

(35) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$ and if $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction Y)$ or $X \subseteq Y$, then $(h\restriction X)\cdot s_1 = (h\restriction Y)\cdot s_1$.

(36) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$, then $|(h\restriction X)\cdot s_1| = (|h|\restriction X)\cdot s_1$.

(37) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$ and $h^{-1}(\{0\}) = \emptyset$, then $(\frac{1}{h}\restriction X)\cdot s_1 = ((h\restriction X)\cdot s_1)^{-1}$.

(38) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $h^\circ \operatorname{rng} s_1 = \operatorname{rng}(h\cdot s_1)$.

(39) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h_2\cdot h_1)$, then $h_2\cdot(h_1\cdot s_1) = (h_2\cdot h_1)\cdot s_1$.

Let Z be a set and let f be an one-to-one function. One can verify that $f\restriction Z$ is one-to-one.

Next we state three propositions:

(40) For every one-to-one function h holds $(h\restriction X)^{-1} = h^{-1}\restriction h^\circ X$.

[2] The proposition (20) has been removed.

(41) If $\operatorname{rng} h$ is bounded and $\sup \operatorname{rng} h = \inf \operatorname{rng} h$, then h is a constant on $\operatorname{dom} h$.

(42) If $Y \subseteq \operatorname{dom} h$ and $h^\circ Y$ is bounded and $\sup(h^\circ Y) = \inf(h^\circ Y)$, then h is a constant on Y.

Let us consider h, Y. We say that h is increasing on Y if and only if:

(Def. 2) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) < h(r_2)$.

We say that h is decreasing on Y if and only if:

(Def. 3) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) < h(r_1)$.

We say that h is non-decreasing on Y if and only if:

(Def. 4) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) \leq h(r_2)$.

We say that h is non increasing on Y if and only if:

(Def. 5) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) \leq h(r_1)$.

Let us consider h, Y. We say that h is monotone on Y if and only if:

(Def. 6) h is non-decreasing on Y and non increasing on Y.

We now state a number of propositions:

(48)[3] h is non-decreasing on Y iff for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_1) \leq h(r_2)$.

(49) h is non increasing on Y iff for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_2) \leq h(r_1)$.

(50) h is increasing on X iff $h \restriction X$ is increasing on X.

(51) h is decreasing on X iff $h \restriction X$ is decreasing on X.

(52) h is non-decreasing on X iff $h \restriction X$ is non-decreasing on X.

(53) h is non increasing on X iff $h \restriction X$ is non increasing on X.

(54) Suppose Y misses $\operatorname{dom} h$. Then h is increasing on Y, decreasing on Y, non-decreasing on Y, non increasing on Y, and monotone on Y.

(55) If h is increasing on Y, then h is non-decreasing on Y.

(56) If h is decreasing on Y, then h is non increasing on Y.

(57) If h is a constant on Y, then h is non-decreasing on Y.

(58) If h is a constant on Y, then h is non increasing on Y.

(59) If h is non-decreasing on Y and non increasing on X, then h is a constant on $Y \cap X$.

(60) If $X \subseteq Y$ and h is increasing on Y, then h is increasing on X.

(61) If $X \subseteq Y$ and h is decreasing on Y, then h is decreasing on X.

(62) If $X \subseteq Y$ and h is non-decreasing on Y, then h is non-decreasing on X.

(63) If $X \subseteq Y$ and h is non increasing on Y, then h is non increasing on X.

(64)(i) If h is increasing on Y and $0 < r$, then $r\,h$ is increasing on Y,

(ii) if $r = 0$, then $r\,h$ is a constant on Y, and

(iii) if h is increasing on Y and $r < 0$, then $r\,h$ is decreasing on Y.

[3] The propositions (43)–(47) have been removed.

(65)(i) If h is decreasing on Y and $0 < r$, then $r\,h$ is decreasing on Y, and

(ii) if h is decreasing on Y and $r < 0$, then $r\,h$ is increasing on Y.

(66)(i) If h is non-decreasing on Y and $0 \leq r$, then $r\,h$ is non-decreasing on Y, and

(ii) if h is non-decreasing on Y and $r \leq 0$, then $r\,h$ is non increasing on Y.

(67)(i) If h is non increasing on Y and $0 \leq r$, then $r\,h$ is non increasing on Y, and

(ii) if h is non increasing on Y and $r \leq 0$, then $r\,h$ is non-decreasing on Y.

(68) If $r \in X \cap Y \cap \mathrm{dom}(h_1 + h_2)$, then $r \in X \cap \mathrm{dom}\, h_1$ and $r \in Y \cap \mathrm{dom}\, h_2$.

(69)(i) If h_1 is increasing on X and h_2 is increasing on Y, then $h_1 + h_2$ is increasing on $X \cap Y$,

(ii) if h_1 is decreasing on X and h_2 is decreasing on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$,

(iii) if h_1 is non-decreasing on X and h_2 is non-decreasing on Y, then $h_1 + h_2$ is non-decreasing on $X \cap Y$, and

(iv) if h_1 is non increasing on X and h_2 is non increasing on Y, then $h_1 + h_2$ is non increasing on $X \cap Y$.

(70)(i) If h_1 is increasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is increasing on $X \cap Y$, and

(ii) if h_1 is decreasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(71) If h_1 is increasing on X and h_2 is non-decreasing on Y, then $h_1 + h_2$ is increasing on $X \cap Y$.

(72) If h_1 is non increasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is non increasing on $X \cap Y$.

(73) If h_1 is decreasing on X and h_2 is non increasing on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(74) If h_1 is non-decreasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is non-decreasing on $X \cap Y$.

(75) h is increasing on $\{x\}$.

(76) h is decreasing on $\{x\}$.

(77) h is non-decreasing on $\{x\}$.

(78) h is non increasing on $\{x\}$.

(79) id_R is increasing on R.

(80) If h is increasing on X, then $-h$ is decreasing on X.

(81) If h is non-decreasing on X, then $-h$ is non increasing on X.

(82) If h is increasing on $[p,g]$ and decreasing on $[p,g]$, then $h \restriction [p,g]$ is one-to-one.

(83) Let h be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. If h is increasing on $[p,g]$, then $(h \restriction [p,g])^{-1}$ is increasing on $h^{\circ}[p,g]$.

(84) Let h be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. If h is decreasing on $[p,g]$, then $(h \restriction [p,g])^{-1}$ is decreasing on $h^{\circ}[p,g]$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[5] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_4.html.

[6] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_2.html.

[7] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seqm_3.html.

[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[9] Jarosław Kotowicz. Partial functions from a domain to a domain. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/partfun2.html.

[10] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_1.html.

[11] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[13] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[16] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received June 18, 1990

Published January 2, 2004