JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Introduction to Theory of Rearrangement[1]

Yuji Sakai
Shinshu University
Nagano

Jarosław Kotowicz
Warsaw University
Białystok

**Summary.** An introduction to the rearrangement theory for finite functions (e.g. with the finite domain and codomain). The notion of generators and cogenerators of finite sets (equivalent to the order in the language of finite sequences) has been defined. The notion of rearrangement for a function into finite set is presented. Some basic properties of these notions have been proved.

MML Identifier: REARRAN1.

WWW: http://mizar.org/JFM/Vol5/rearran1.html

The articles [16], [6], [19], [17], [20], [4], [3], [1], [9], [11], [2], [18], [21], [5], [12], [13], [7], [8], [10], [14], and [15] provide the notation and terminology for this paper.

In this paper $n$, $m$ denote natural numbers and $r$ denotes a real number.

Let $D$ be a non empty set, let $F$ be a partial function from $D$ to $\mathbb{R}$, and let $r$ be a real number. Then $r\,F$ is an element of $D \dot{\rightarrow} \mathbb{R}$.

Let $I_1$ be a finite sequence. We say that $I_1$ has cardinality by index if and only if:

(Def. 1) For every $n$ such that $1 \leq n$ and $n \leq \operatorname{len} I_1$ and for every finite set $B$ such that $B = I_1(n)$ holds $\operatorname{card} B = n$.

We say that $I_1$ is ascending if and only if:

(Def. 2) For every $n$ such that $1 \leq n$ and $n \leq \operatorname{len} I_1 - 1$ holds $I_1(n) \subseteq I_1(n+1)$.

Let $X$ be a set and let $I_1$ be a finite sequence of elements of $X$. We say that $I_1$ has length by cardinality if and only if:

(Def. 3) There exists a finite set $B$ such that $B = \bigcup X$ and $\operatorname{len} I_1 = \operatorname{card} B$.

Let $D$ be a non empty finite set. One can check that there exists a finite sequence of elements of $2^D$ which is ascending and has cardinality by index and length by cardinality.

Let $D$ be a non empty finite set. A rearrangement generator of $D$ is an ascending finite sequence of elements of $2^D$ with cardinality by index and length by cardinality.

In the sequel $C$, $D$ denote non empty finite sets and $a$ denotes a finite sequence of elements of $2^D$.

We now state a number of propositions:

(1) For every finite sequence $a$ of elements of $2^D$ holds $a$ has length by cardinality iff $\operatorname{len} a = \operatorname{card} D$.

(2) Let $a$ be a finite sequence. Then $a$ is ascending if and only if for all $n$, $m$ such that $n \leq m$ and $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ holds $a(n) \subseteq a(m)$.

[1]Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday.

(3) For every finite sequence $a$ of elements of $2^D$ with cardinality by index and length by cardinality holds $a(\operatorname{len} a) = D$.

(4) For every finite sequence $a$ of elements of $2^D$ with length by cardinality holds $\operatorname{len} a \neq 0$.

(5) Let $a$ be an ascending finite sequence of elements of $2^D$ with cardinality by index and given $n$, $m$. If $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ and $n \neq m$, then $a(n) \neq a(m)$.

(6) Let $a$ be an ascending finite sequence of elements of $2^D$ with cardinality by index and given $n$. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n) \neq a(n+1)$.

(7) For every finite sequence $a$ of elements of $2^D$ with cardinality by index such that $n \in \operatorname{dom} a$ holds $a(n) \neq \emptyset$.

(8) Let $a$ be a finite sequence of elements of $2^D$ with cardinality by index. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n+1) \setminus a(n) \neq \emptyset$.

(9) Let $a$ be a finite sequence of elements of $2^D$ with cardinality by index and length by cardinality. Then there exists an element $d$ of $D$ such that $a(1) = \{d\}$.

(10) Let $a$ be an ascending finite sequence of elements of $2^D$ with cardinality by index. Suppose $1 \leq n$ and $n \leq \operatorname{len} a - 1$. Then there exists an element $d$ of $D$ such that $a(n+1) \setminus a(n) = \{d\}$ and $a(n+1) = a(n) \cup \{d\}$ and $a(n+1) \setminus \{d\} = a(n)$.

Let $D$ be a non empty finite set and let $A$ be a rearrangement generator of $D$. The functor co-Gen($A$) yields a rearrangement generator of $D$ and is defined as follows:

(Def. 4) For every $m$ such that $1 \leq m$ and $m \leq \operatorname{len}\text{co-Gen}(A) - 1$ holds $(\text{co-Gen}(A))(m) = D \setminus A(\operatorname{len} A - m)$.

Next we state two propositions:

(11) For every rearrangement generator $A$ of $D$ holds $\text{co-Gen}(\text{co-Gen}(A)) = A$.

(12) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{len} \mathrm{MIM}(\mathrm{FinS}(F,D)) = \operatorname{len} \mathrm{CHI}(A,C)$.

Let $D$, $C$ be non empty finite sets, let $A$ be a rearrangement generator of $C$, and let $F$ be a partial function from $D$ to $\mathbb{R}$. The functor $F_A^{\wedge}$ yielding a partial function from $C$ to $\mathbb{R}$ is defined as follows:

(Def. 5) $F_A^{\wedge} = \sum(\mathrm{MIM}(\mathrm{FinS}(F,D))\,\mathrm{CHI}(A,C))$.

The functor $F_A^{\vee}$ yields a partial function from $C$ to $\mathbb{R}$ and is defined as follows:

(Def. 6) $F_A^{\vee} = \sum(\mathrm{MIM}(\mathrm{FinS}(F,D))\,\mathrm{CHI}(\text{co-Gen}(A),C))$.

One can prove the following propositions:

(13) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F_A^{\wedge} = C$.

(14) Let $c$ be an element of $C$, $F$ be a partial function from $D$ to $\mathbb{R}$, and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then

(i) if $c \in A(1)$, then $(\mathrm{MIM}(\mathrm{FinS}(F,D))\,\mathrm{CHI}(A,C))\#c = \mathrm{MIM}(\mathrm{FinS}(F,D))$, and

(ii) for every $n$ such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(\mathrm{MIM}(\mathrm{FinS}(F,D))\,\mathrm{CHI}(A,C))\#c = (n \mapsto (0$ **qua** real number$))^\frown \mathrm{MIM}((\mathrm{FinS}(F,D))_{\restriction n})$.

(15) Let $c$ be an element of $C$, $F$ be a partial function from $D$ to $\mathbb{R}$, and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in A(1)$, then $(F_A^{\wedge})(c) = (\mathrm{FinS}(F,D))(1)$ and for every $n$ such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(F_A^{\wedge})(c) = (\mathrm{FinS}(F,D))(n+1)$.

(16) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F_A^\wedge = \operatorname{rng} \operatorname{FinS}(F, D)$.

(17) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^\wedge$ and $\operatorname{FinS}(F, D)$ are fiberwise equipotent.

(18) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}(F_A^\wedge, C) = \operatorname{FinS}(F, D)$.

(19) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^\wedge(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(20) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}((F_A^\wedge) - r, C) = \operatorname{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C} ((F_A^\wedge) - r)(\kappa) = \sum_{\kappa=0}^{D} (F - r)(\kappa)$.

(21) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F_A^\vee = C$.

(22) Let $c$ be an element of $C$, $F$ be a partial function from $D$ to $\mathbb{R}$, and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in (\text{co-Gen}(A))(1)$, then $(F_A^\vee)(c) = (\operatorname{FinS}(F, D))(1)$ and for every $n$ such that $1 \leq n$ and $n < \operatorname{len} \text{co-Gen}(A)$ and $c \in (\text{co-Gen}(A))(n+1) \setminus (\text{co-Gen}(A))(n)$ holds $(F_A^\vee)(c) = (\operatorname{FinS}(F, D))(n+1)$.

(23) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F_A^\vee = \operatorname{rng} \operatorname{FinS}(F, D)$.

(24) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^\vee$ and $\operatorname{FinS}(F, D)$ are fiberwise equipotent.

(25) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}(F_A^\vee, C) = \operatorname{FinS}(F, D)$.

(26) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^\vee(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(27) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}((F_A^\vee) - r, C) = \operatorname{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C} ((F_A^\vee) - r)(\kappa) = \sum_{\kappa=0}^{D} (F - r)(\kappa)$.

(28) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^\vee$ and $F_A^\wedge$ are fiberwise equipotent and $\operatorname{FinS}(F_A^\vee, C) = \operatorname{FinS}(F_A^\wedge, C)$ and $\sum_{\kappa=0}^{C} F_A^\vee(\kappa) = \sum_{\kappa=0}^{C} F_A^\wedge(\kappa)$.

(29) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_+((F_A^\wedge) - r)$ and $\max_+(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_+((F_A^\wedge) - r), C) = \operatorname{FinS}(\max_+(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_+((F_A^\wedge) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_+(F - r)(\kappa)$.

(30) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_-((F_A^\wedge) - r)$ and $\max_-(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_-((F_A^\wedge) - r), C) = \operatorname{FinS}(\max_-(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_-((F_A^\wedge) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_-(F - r)(\kappa)$.

(31) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \operatorname{FinS}(F_A^\wedge, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \operatorname{FinS}(F_A^\wedge, C)$.

(32) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\operatorname{FinS}(F_A^\wedge, C) \restriction n = \operatorname{FinS}(F_A^\wedge, A(n))$.

(33) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^\wedge = (F_A^\wedge) - r$.

(34) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_+((F_A^\vee) - r)$ and $\max_+(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_+((F_A^\vee) - r), C) = \operatorname{FinS}(\max_+(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_+((F_A^\vee) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_+(F - r)(\kappa)$.

(35) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_-((F_A^\vee) - r)$ and $\max_-(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_-((F_A^\vee) - r), C) = \operatorname{FinS}(\max_-(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_-((F_A^\vee) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_-(F - r)(\kappa)$.

(36) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \operatorname{FinS}(F_A^\vee, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \operatorname{FinS}(F_A^\vee, C)$.

(37) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\operatorname{FinS}(F_A^\vee, C) \restriction n = \operatorname{FinS}(F_A^\vee, (\text{co-Gen}(A))(n))$.

(38) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^\vee = (F_A^\vee) - r$.

(39) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} D = \operatorname{card} C$. Then $F_A^\wedge$ and $F$ are fiberwise equipotent and $F_A^\vee$ and $F$ are fiberwise equipotent and $\operatorname{rng} F_A^\wedge = \operatorname{rng} F$ and $\operatorname{rng} F_A^\vee = \operatorname{rng} F$.

## REFERENCES


[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[7] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseqop.html.

[8] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.

[9] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[10] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}_T^2$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[13] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_1.html.

[14] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfinseq.html.

[15] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfunct_3.html.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[17] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[19] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[21] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

*Received May 22, 1993*

*Published January 2, 2004*