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Properties of Fields

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Summary. The second part of considerations concerning groups and fields. It includes a definition and properties of commutative field F as a structure defined by: the set, a support of F, containing two different elements, by two binary operations $+_F$, $\cdot_F$ on this set, called addition and multiplication, and by two elements from the support of F, $\mathbf{0}_F$ being neutral for addition and $\mathbf{1}_F$ being neutral for multiplication. This structure is named a field if $\langle$the support of F, $+_F$, $\mathbf{0}_F\rangle$ and $\langle$the support of F, $\cdot_F$, $\mathbf{1}_F\rangle$ are commutative groups and multiplication has the property of left-hand and right-hand distributivity with respect to addition. It is demonstrated that the field F satisfies the definition of a field in the axiomatic approach.

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The articles [6], [4], [8], [9], [2], [3], [7], [5], and [1] provide the notation and terminology for this paper.

Let I_1 be a double loop structure. We say that I_1 is field-like if and only if the condition (Def. 1) is satisfied.

(Def. 1) There exists a non trivial set A and there exists a binary operation o_1 on A and there exists an element n_1 of A and there exists a binary operation o_2 of A preserving $A \setminus \{ n_1 \}$ and there exists an element n_2 of $A \setminus \{n_1\}$ such that

(i) $I_1 = \mathrm{field}(A, o_1, o_2, n_1, n_2)$,

(ii) $\langle A, o_1, n_1 \rangle$ is a group,

(iii) for every non empty set B and for every binary operation P on B and for every element e of B such that $B = A \setminus \{n_1\}$ and $e = n_2$ and $P = o_2 \restriction_{n_1} A$ holds $\langle B, P, e \rangle$ is a group, and

(iv) for all elements x, y, z of A holds $o_2(\langle x, o_1(\langle y, z \rangle)\rangle) = o_1(\langle o_2(\langle x, y \rangle), o_2(\langle x, z \rangle)\rangle)$ and $o_2(\langle o_1(\langle x, y \rangle), z \rangle) = o_1(\langle o_2(\langle x, z \rangle), o_2(\langle y, z \rangle)\rangle)$.

One can check that there exists a double loop structure which is strict and field-like.

A field is a field-like double loop structure.

Let F be a field. The support of F yields a non trivial set and is defined by the condition (Def. 2).

(Def. 2) There exists a binary operation o_1 on the support of F and there exists an element n_1 of the support of F and there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{ n_1 \}$ and there exists an element n_2 of (the support of F) $\setminus \{n_1\}$ such that $F = \mathrm{field}(\text{the support of } F, o_1, o_2, n_1, n_2)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x \rangle, x\rangle$, $x_2 = \langle\langle x, y \rangle, y\rangle$, $x_3 = \langle\langle y, x \rangle, y\rangle$, and $x_4 = \langle\langle y, y \rangle, x\rangle$.

Let F be a field. The functor $+_F$ yielding a binary operation on the support of F is defined by the condition (Def. 3).

(Def. 3) There exists an element n_1 of the support of F and there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{ n_1 \}$ and there exists an element n_2 of (the support of $F) \setminus \{n_1\}$ such that $F =$ field(the support of F, $+_F, o_2, n_1, n_2)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a field. The functor $\mathbf{0}_F$ yielding an element of the support of F is defined by the condition (Def. 4).

(Def. 4) There exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{ \mathbf{0}_F \}$ and there exists an element n_2 of (the support of $F) \setminus \{\mathbf{0}_F\}$ such that $F =$ field(the support of F, $+_F, o_2, \mathbf{0}_F, n_2)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a field. The functor $\cdot_F$ yields a binary operation of the support of F preserving the support of $F \setminus \{ \mathbf{0}_F \}$ and is defined by:

(Def. 5) There exists an element n_2 of (the support of $F) \setminus \{\mathbf{0}_F\}$ such that $F =$ field(the support of F, $+_F, \cdot_F, \mathbf{0}_F, n_2)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a field. The functor $\mathbf{1}_F$ yields an element of (the support of $F) \setminus \{\mathbf{0}_F\}$ and is defined as follows:

(Def. 6) $F =$ field(the support of F, $+_F, \cdot_F, \mathbf{0}_F, \mathbf{1}_F)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

We now state several propositions:

(8)[1] For every field F holds $\langle$the support of F, $+_F, \mathbf{0}_F\rangle$ is a group, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(9) Let F be a field, B be a non empty set, P be a binary operation on B, and e be an element of B. Suppose $B =$ (the support of $F) \setminus \{\mathbf{0}_F\}$ and $e = \mathbf{1}_F$ and $P = \cdot_F \restriction_{\mathbf{0}_F}$ the support of F. Then $\langle B, P, e\rangle$ is a group, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(10) Let F be a field and x, y, z be elements of the support of F. Then $\cdot_F(\langle x, +_F(\langle y, z\rangle)\rangle) = +_F(\langle \cdot_F(\langle x, y\rangle), \cdot_F(\langle x, z\rangle)\rangle)$ and $\cdot_F(\langle +_F(\langle x, y\rangle), z\rangle) = +_F(\langle \cdot_F(\langle x, z\rangle), \cdot_F(\langle y, z\rangle)\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(11) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle +_F(\langle a, b\rangle), c\rangle) = +_F(\langle a, +_F(\langle b, c\rangle)\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(12) For every field F and for all elements a, b of the support of F holds $+_F(\langle a, b\rangle) = +_F(\langle b, a\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(13) For every field F and for every element a of the support of F holds $+_F(\langle a, \mathbf{0}_F\rangle) = a$ and $+_F(\langle \mathbf{0}_F, a\rangle) = a$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(14) Let F be a field and a be an element of the support of F. Then there exists an element b of the support of F such that $+_F(\langle a, b\rangle) = \mathbf{0}_F$ and $+_F(\langle b, a\rangle) = \mathbf{0}_F$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a non trivial set. A set is called an one-element subset of F if:

[1] The propositions (1)–(7) have been removed.

(Def. 7) There exists an element x of F such that it $= \{x\}$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Next we state the proposition

(15) For every non trivial set F and for every one-element subset A of F holds $F \setminus A$ is a non empty set, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a non trivial set and let A be an one-element subset of F. Note that $F \setminus A$ is non empty.

Let F be a non trivial set. One can check that there exists an one-element subset of F which is non empty.

Let F be a non trivial set and let x be an element of F. Then $\{x\}$ is an one-element subset of F.

Next we state four propositions:

(20)[2] For every field F and for all elements a, b, c of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $\cdot_F(\langle\cdot_F(\langle a, b\rangle), c\rangle) = \cdot_F(\langle a, \cdot_F(\langle b, c\rangle)\rangle)$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(21) For every field F and for all elements a, b of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $\cdot_F(\langle a, b\rangle) = \cdot_F(\langle b, a\rangle)$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(22) For every field F and for every element a of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $\cdot_F(\langle a, \mathbf{1}_F\rangle) = a$ and $\cdot_F(\langle \mathbf{1}_F, a\rangle) = a$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(23) Let F be a field and a be an element of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then there exists an element b of (the support of F) $\setminus \{\mathbf{0}_F\}$ such that $\cdot_F(\langle a, b\rangle) = \mathbf{1}_F$ and $\cdot_F(\langle b, a\rangle) = \mathbf{1}_F$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a field. The functor $-_F$ yielding a function from the support of F into the support of F is defined as follows:

(Def. 8) For every element x of the support of F holds $+_F(\langle x, -_F(x)\rangle) = \mathbf{0}_F$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

One can prove the following propositions:

(26)[3] For every field F and for all elements x, y of the support of F such that $+_F(\langle x, y\rangle) = \mathbf{0}_F$ holds $y = -_F(x)$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(27) For every field F and for every element x of the support of F holds $x = -_F(-_F(x))$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(28) Let F be a field and a, b be elements of the support of F. Then

(i) $+_F(\langle a, b\rangle)$ is an element of the support of F,

(ii) $\cdot_F(\langle a, b\rangle)$ is an element of the support of F, and

(iii) $-_F(a)$ is an element of the support of F,

where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(29) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle a, +_F(\langle b, -_F(c)\rangle)\rangle) = +_F(\langle\cdot_F(\langle a, b\rangle), -_F(\cdot_F(\langle a, c\rangle))\rangle)$, where $F = \{x_1,x_2,x_3,x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

[2] The propositions (16)–(19) have been removed.

[3] The propositions (24) and (25) have been removed.

(30) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle +_F(\langle a, -_F(b)\rangle), c\rangle) = +_F(\langle \cdot_F(\langle a, c\rangle), -_F(\cdot_F(\langle b, c\rangle))\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(31) For every field F and for every element a of the support of F holds $\cdot_F(\langle a, \mathbf{0}_F\rangle) = \mathbf{0}_F$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(32) For every field F and for every element a of the support of F holds $\cdot_F(\langle \mathbf{0}_F, a\rangle) = \mathbf{0}_F$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(33) For every field F and for all elements a, b of the support of F holds $-_F(\cdot_F(\langle a, b\rangle)) = \cdot_F(\langle a, -_F(b)\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(34) For every field F holds $\cdot_F(\langle \mathbf{1}_F, \mathbf{0}_F\rangle) = \mathbf{0}_F$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(35) For every field F holds $\cdot_F(\langle \mathbf{0}_F, \mathbf{1}_F\rangle) = \mathbf{0}_F$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(36) Let F be a field and a, b be elements of the support of F. Then $\cdot_F(\langle a, b\rangle)$ is an element of the support of F, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(37) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle \cdot_F(\langle a, b\rangle), c\rangle) = \cdot_F(\langle a, \cdot_F(\langle b, c\rangle)\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(38) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle) = \cdot_F(\langle b, a\rangle)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(39) For every field F and for every element a of the support of F holds $\cdot_F(\langle a, \mathbf{1}_F\rangle) = a$ and $\cdot_F(\langle \mathbf{1}_F, a\rangle) = a$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

Let F be a field. The functor $\overset{-1}{F}$ yields a function from (the support of F) $\setminus \{\mathbf{0}_F\}$ into (the support of F) $\setminus \{\mathbf{0}_F\}$ and is defined as follows:

(Def. 9) For every element x of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $\cdot_F(\langle x, (\overset{-1}{F})(x)\rangle) = \mathbf{1}_F$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

One can prove the following propositions:

(42)[4] For every field F and for all elements x, y of (the support of F) $\setminus \{\mathbf{0}_F\}$ such that $\cdot_F(\langle x, y\rangle) = \mathbf{1}_F$ holds $y = (\overset{-1}{F})(x)$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(43) For every field F and for every element x of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $x = (\overset{-1}{F})((\overset{-1}{F})(x))$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(44) Let F be a field and a, b be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, b\rangle)$ is an element of (the support of F) $\setminus \{\mathbf{0}_F\}$ and $(\overset{-1}{F})(a)$ is an element of (the support of F) $\setminus \{\mathbf{0}_F\}$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

(45) For every field F and for all elements a, b, c of the support of F such that $+_F(\langle a, b\rangle) = +_F(\langle a, c\rangle)$ holds $b = c$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

[4] The propositions (40) and (41) have been removed.

(46) Let F be a field, a be an element of (the support of F) $\setminus \{\mathbf{0}_F\}$, and b, c be elements of the support of F. If $\cdot_F(\langle a, b\rangle) = \cdot_F(\langle a, c\rangle)$, then $b = c$, where $F = \{x_1, x_2, x_3, x_4\}$, $x_1 = \langle\langle x, x\rangle, x\rangle$, $x_2 = \langle\langle x, y\rangle, y\rangle$, $x_3 = \langle\langle y, x\rangle, y\rangle$, and $x_4 = \langle\langle y, y\rangle, x\rangle$.

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