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Equalities and Inequalities in Real Numbers

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Summary. The aim of the article is to give a number of useful theorems concerning equalities and inequalities in real numbers. Some of the theorems are extensions of [2] theorems, others were found to be needed in practice.

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The articles [4], [1], [3], and [2] provide the notation and terminology for this paper.

In this paper a, b, d, e are real numbers.

The following propositions are true:

(63)[1] For all a, b there exists e such that $a = b - e$.

(64) For all a, b such that $a \neq 0$ and $b \neq 0$ there exists e such that $a = \frac{b}{e}$.

(105)[2] If $a + -b \leq 0$ or $b - a \geq 0$ or $b + -a \geq 0$ or $a - e \leq b + -e$ or $a + -e \leq b - e$ or $e - a \geq e - b$, then $a \leq b$.

(106) If $a + -b < 0$ or $b - a > 0$ or $-a + b > 0$ or $a - e < b + -e$ or $a + -e < b - e$ or $e - a > e - b$, then $a < b$.

(109)[3] If $a \leq -b$, then $a + b \leq 0$ and $-a \geq b$.

(110) If $a < -b$, then $a + b < 0$ and $-a > b$.

(111) If $-a \leq b$, then $a + b \geq 0$.

(112) If $-b < a$, then $a + b > 0$.

(117)[4] Suppose $b > 0$. Then

(i) if $\frac{a}{b} > 1$, then $a > b$,

(ii) if $\frac{a}{b} < 1$, then $a < b$,

(iii) if $\frac{a}{b} > -1$, then $a > -b$ and $b > -a$, and

(iv) if $\frac{a}{b} < -1$, then $a < -b$ and $b < -a$.

[1] The propositions (1)–(62) have been removed.
[2] The propositions (65)–(104) have been removed.
[3] The propositions (107) and (108) have been removed.
[4] The propositions (113)–(116) have been removed.

(118) Suppose $b > 0$. Then

(i) if $\frac{a}{b} \geq 1$, then $a \geq b$,

(ii) if $\frac{a}{b} \leq 1$, then $a \leq b$,

(iii) if $\frac{a}{b} \geq -1$, then $a \geq -b$ and $b \geq -a$, and

(iv) if $\frac{a}{b} \leq -1$, then $a \leq -b$ and $b \leq -a$.

(119) Suppose $b < 0$. Then

(i) if $\frac{a}{b} > 1$, then $a < b$,

(ii) if $\frac{a}{b} < 1$, then $a > b$,

(iii) if $\frac{a}{b} > -1$, then $a < -b$ and $b < -a$, and

(iv) if $\frac{a}{b} < -1$, then $a > -b$ and $b > -a$.

(120) Suppose $b < 0$. Then

(i) if $\frac{a}{b} \geq 1$, then $a \leq b$,

(ii) if $\frac{a}{b} \leq 1$, then $a \geq b$,

(iii) if $\frac{a}{b} \geq -1$, then $a \leq -b$ and $b \leq -a$, and

(iv) if $\frac{a}{b} \leq -1$, then $a \geq -b$ and $b \geq -a$.

(121) If $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$, then $a \cdot b \geq 0$.

(122) If $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$, then $a \cdot b > 0$.

(123) If $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$, then $a \cdot b \leq 0$.

(125)[5] If $a \leq 0$ and $b \leq 0$ or $a \geq 0$ and $b \geq 0$, then $\frac{a}{b} \geq 0$.

(126) If $a \geq 0$ and $b < 0$ or $a \leq 0$ and $b > 0$, then $\frac{a}{b} \leq 0$.

(127) If $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$.

(128) If $a < 0$ and $b > 0$, then $\frac{a}{b} < 0$ and $\frac{b}{a} < 0$.

(129) If $a \cdot b \leq 0$, then $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$.

(132)[6] If $a \cdot b < 0$, then $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

(133) If $b \neq 0$ and $\frac{a}{b} \leq 0$, then $b > 0$ and $a \leq 0$ or $b < 0$ and $a \geq 0$.

(134) If $b \neq 0$ and $\frac{a}{b} \geq 0$, then $b > 0$ and $a \geq 0$ or $b < 0$ and $a \leq 0$.

(135) If $b \neq 0$ and $\frac{a}{b} < 0$, then $b < 0$ and $a > 0$ or $b > 0$ and $a < 0$.

(136) If $b \neq 0$ and $\frac{a}{b} > 0$, then $b > 0$ and $a > 0$ or $b < 0$ and $a < 0$.

(137) If $a > 1$ but $b > 1$ or $b \geq 1$ or $a < -1$ but $b < -1$ or $b \leq -1$, then $a \cdot b > 1$.

(138) If $a \geq 1$ and $b \geq 1$ or $a \leq -1$ and $b \leq -1$, then $a \cdot b \geq 1$.

(139) If $0 \leq a$ and $a < 1$ and $0 \leq b$ and $b \leq 1$ or $0 \geq a$ and $a > -1$ and $0 \geq b$ and $b \geq -1$, then $a \cdot b < 1$.

(140) If $0 \leq a$ and $a \leq 1$ and $0 \leq b$ and $b \leq 1$ or $0 \geq a$ and $a \geq -1$ and $0 \geq b$ and $b \geq -1$, then $a \cdot b \leq 1$.

(142)[7] If $0 < a$ and $a < b$ or $b < a$ and $a < 0$, then $\frac{a}{b} < 1$ and $\frac{b}{a} > 1$.

[5] The proposition (124) has been removed.
[6] The propositions (130) and (131) have been removed.
[7] The proposition (141) has been removed.

(143) If $0 < a$ and $a \leq b$ or $b \leq a$ and $a < 0$, then $\frac{a}{b} \leq 1$ and $\frac{b}{a} \geq 1$.

(144) If $a > 0$ and $b > 1$ or $a < 0$ and $b < 1$, then $a \cdot b > a$.

(145) If $a > 0$ and $b < 1$ or $a < 0$ and $b > 1$, then $a \cdot b < a$.

(146) If $a \geq 0$ and $b \geq 1$ or $a \leq 0$ and $b \leq 1$, then $a \cdot b \geq a$.

(147) If $a \geq 0$ and $b \leq 1$ or $a \leq 0$ and $b \geq 1$, then $a \cdot b \leq a$.

(149)[8] If $a < 0$, then $\frac{1}{a} < 0$ and $a^{-1} < 0$.

(150) If $\frac{1}{a} < 0$, then $a < 0$ and if $\frac{1}{a} > 0$, then $a > 0$.

(151) If $0 < a$ or $b < 0$ and if $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(152) If $0 < a$ or $b < 0$ and if $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.

(153) If $a < 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$.

(154) If $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and if $\frac{1}{a} > \frac{1}{b}$, then $a < b$.

(155) If $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and if $\frac{1}{a} \geq \frac{1}{b}$, then $a \leq b$.

(156) If $\frac{1}{a} < 0$ and $\frac{1}{b} > 0$, then $a < b$.

(157) If $a < -1$, then $\frac{1}{a} > -1$.

(158) If $a \leq -1$, then $\frac{1}{a} \geq -1$.

(164)[9] If $1 \leq a$, then $\frac{1}{a} \leq 1$.

(165) If $b \leq e - a$, then $a \leq e - b$ and if $b \geq e - a$, then $a \geq e - b$.

(167)[10] If $a + b \leq e + d$, then $a - e \leq d - b$.

(168) If $a + b < e + d$, then $a - e < d - b$.

(169) If $a - b \leq e - d$, then $a + d \leq e + b$ and $a - e \leq b - d$ and $e - a \geq d - b$ and $b - a \geq d - e$.

(170) If $a - b < e - d$, then $a + d < e + b$ and $a - e < b - d$ and $e - a > d - b$ and $b - a > d - e$.

(171) If $a + b \leq e - d$, then $a + d \leq e - b$ and if $a + b \geq e - d$, then $a + d \geq e - b$.

(173)[11] If $a < 0$, then $a + b < b$ and $b - a > b$ and if $a + b < b$ or $b - a > b$, then $a < 0$.

(174) If $a \leq 0$, then $a + b \leq b$ and $b - a \geq b$ and if $a + b \leq b$ or $b - a \geq b$, then $a \leq 0$.

(177)[12](i) If $b > 0$ and $a \cdot b \leq e$, then $a \leq \frac{e}{b}$,
(ii) if $b < 0$ and $a \cdot b \leq e$, then $a \geq \frac{e}{b}$,
(iii) if $b > 0$ and $a \cdot b \geq e$, then $a \geq \frac{e}{b}$, and
(iv) if $b < 0$ and $a \cdot b \geq e$, then $a \leq \frac{e}{b}$.

(178)(i) If $b > 0$ and $a \cdot b < e$, then $a < \frac{e}{b}$,
(ii) if $b < 0$ and $a \cdot b < e$, then $a > \frac{e}{b}$,
(iii) if $b > 0$ and $a \cdot b > e$, then $a > \frac{e}{b}$, and
(iv) if $b < 0$ and $a \cdot b > e$, then $a < \frac{e}{b}$.

[8] The proposition (148) has been removed.
[9] The propositions (159)–(163) have been removed.
[10] The proposition (166) has been removed.
[11] The proposition (172) has been removed.
[12] The propositions (175) and (176) have been removed.

(181)[13] If for every a such that $a > 0$ holds $b + a \geq e$ or for every a such that $a < 0$ holds $b - a \geq e$, then $b \geq e$.

(182) If for every a such that $a > 0$ holds $b - a \leq e$ or for every a such that $a < 0$ holds $b + a \leq e$, then $b \leq e$.

(183) If for every a such that $a > 1$ holds $b \cdot a \geq e$ or for every a such that $0 < a$ and $a < 1$ holds $\frac{b}{a} \geq e$, then $b \geq e$.

(184) If for every a such that $0 < a$ and $a < 1$ holds $b \cdot a \leq e$ or for every a such that $a > 1$ holds $\frac{b}{a} \leq e$, then $b \leq e$.

(185) If $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ and if $a \cdot d < e \cdot b$, then $\frac{a}{b} < \frac{e}{d}$.

(186) If $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ and if $a \cdot d < e \cdot b$, then $\frac{a}{b} > \frac{e}{d}$.

(187) If $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ and if $a \cdot d \leq e \cdot b$, then $\frac{a}{b} \leq \frac{e}{d}$.

(188) If $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ and if $a \cdot d \leq e \cdot b$, then $\frac{a}{b} \geq \frac{e}{d}$.

(193)[14] If $b < 0$ and $d < 0$ or $b > 0$ and $d > 0$, then if $a \cdot b < \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$ and if $a \cdot b > \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$.

(194) If $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$, then if $a \cdot b < \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$ and if $a \cdot b > \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$.

(197)[15] If $0 < a$ or $0 \leq a$ and $a < b$ or $a \leq b$ and $0 < e$ or $0 \leq e$ and $e \leq d$, then $a \cdot e \leq b \cdot d$.

(198) If $0 \geq a$ and $a \geq b$ and $0 \geq e$ and $e \geq d$, then $a \cdot e \leq b \cdot d$.

(199) If $0 < a$ and $a \leq b$ and $0 < e$ and $e < d$ or $0 > a$ and $a \geq b$ and $0 > e$ and $e > d$, then $a \cdot e < b \cdot d$.

(200) If $e > 0$ and $a > 0$ and $a < b$, then $\frac{e}{a} > \frac{e}{b}$ and if $e > 0$ and $b < 0$ and $a < b$, then $\frac{e}{a} > \frac{e}{b}$.

(201) If $e \geq 0$ and if $a > 0$ or $b < 0$ and if $a \leq b$, then $\frac{e}{a} \geq \frac{e}{b}$.

(202) If $e < 0$ and if $a > 0$ or $b < 0$ and if $a < b$, then $\frac{e}{a} < \frac{e}{b}$.

(203) If $e \leq 0$ and if $a > 0$ or $b < 0$ and if $a \leq b$, then $\frac{e}{a} \leq \frac{e}{b}$.

(204) Let X, Y be subsets of $\mathbb{R}$. Suppose $X \neq \emptyset$ and $Y \neq \emptyset$ and for all a, b such that $a \in X$ and $b \in Y$ holds $a \leq b$. Then there exists d such that for every a such that $a \in X$ holds $a \leq d$ and for every b such that $b \in Y$ holds $d \leq b$.

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[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[3] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[13] The propositions (179) and (180) have been removed.
[14] The propositions (189)–(192) have been removed.
[15] The propositions (195) and (196) have been removed.

[4] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

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