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Topological Properties of Subsets in Real Numbers[1]

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Summary. The following notions for real subsets are defined: open set, closed set, compact set, intervals and neighbourhoods. In the sequel some theorems involving above mentioned notions are proved.

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The articles [8], [10], [1], [9], [11], [2], [6], [4], [5], [3], and [7] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: n, m are natural numbers, s, g, g_1, g_2, r, p, q are real numbers, s_1, s_2 are sequences of real numbers, and X, Y, Y_1 are subsets of $\mathbb{R}$.

The scheme *RealSeqChoice* concerns a binary predicate $\mathcal{P}$, and states that:

There exists a function s_1 from $\mathbb{N}$ into $\mathbb{R}$ such that for every natural number n holds $\mathcal{P}[n, s_1(n)]$

provided the parameters satisfy the following condition:

- For every natural number n there exists a real number r such that $\mathcal{P}[n, r]$.

We now state four propositions:

(1) If for every r such that $r \in X$ holds $r \in Y$, then $X \subseteq Y$.

(3)[1] If $Y_1 \subseteq Y$ and Y is lower bounded, then Y_1 is lower bounded.

(4) If $Y_1 \subseteq Y$ and Y is upper bounded, then Y_1 is upper bounded.

(5) If $Y_1 \subseteq Y$ and Y is bounded, then Y_1 is bounded.

Let g, s be real numbers. The functor $[g, s]$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 1) $[g, s] = \{r; r$ ranges over real numbers: $g \leq r \wedge r \leq s\}$.

Let g, s be real numbers. The functor $]g, s[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 2) $]g, s[= \{r; r$ ranges over real numbers: $g < r \wedge r < s\}$.

Next we state a number of propositions:

(8)[2] $r \in]p - g, p + g[$ iff $|r - p| < g$.

(9) $r \in [p, g]$ iff $|(p + g) - 2 \cdot r| \leq g - p$.

[1] Supported by RPBP.III-24.C8.

[1] The proposition (2) has been removed.

[2] The propositions (6) and (7) have been removed.

(10) $r \in]p,g[$ iff $|(p+g)-2\cdot r| < g-p$.

(11) For all g, s such that $g \leq s$ holds $[g,s] =]g,s[\cup \{g,s\}$.

(12) If $p \leq g$, then $]g,p[= \emptyset$.

(13) If $p < g$, then $[g,p] = \emptyset$.

(14) $[p,p] = \{p\}$.

(15) If $p < g$, then $]p,g[\neq \emptyset$ and if $p \leq g$, then $p \in [p,g]$ and $g \in [p,g]$ and $]p,g[\subseteq [p,g]$.

(16) If $r \in [p,g]$ and $s \in [p,g]$, then $[r,s] \subseteq [p,g]$.

(17) If $r \in]p,g[$ and $s \in]p,g[$, then $[r,s] \subseteq]p,g[$.

(18) If $p \leq g$, then $[p,g] = [p,g] \cup [g,p]$.

Let us consider X. We say that X is compact if and only if:

(Def. 3) For every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and convergent and $\lim s_2 \in X$.

Let us consider X. We say that X is closed if and only if:

(Def. 4) For every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

Let us consider X. We say that X is open if and only if:

(Def. 5) X^{c} is closed.

One can prove the following four propositions:

(22)[3] For every s_1 such that $\operatorname{rng} s_1 \subseteq [s,g]$ holds s_1 is bounded.

(23) $[s,g]$ is closed.

(24) $[s,g]$ is compact.

(25) $]p,q[$ is open.

Let p, q be real numbers. Note that $]p,q[$ is open.
We now state several propositions:

(26) If X is compact, then X is closed.

(27) Suppose that for every p such that $p \in X$ there exist r, n such that $0 < r$ and for every m such that $n < m$ holds $r < |s_1(m) - p|$. Let given s_2. If s_2 is a subsequence of s_1, then s_2 is not convergent or $\lim s_2 \notin X$.

(28) If X is compact, then X is bounded.

(29) If X is bounded and closed, then X is compact.

(30) For every X such that $X \neq \emptyset$ and X is closed and upper bounded holds $\sup X \in X$.

(31) For every X such that $X \neq \emptyset$ and X is closed and lower bounded holds $\inf X \in X$.

(32) For every X such that $X \neq \emptyset$ and X is compact holds $\sup X \in X$ and $\inf X \in X$.

(33) If X is compact and for all g_1, g_2 such that $g_1 \in X$ and $g_2 \in X$ holds $[g_1, g_2] \subseteq X$, then there exist p, g such that $X = [p,g]$.

[3] The propositions (19)–(21) have been removed.

Let us observe that there exists a subset of $\mathbb{R}$ which is open.

Let r be a real number. A subset of $\mathbb{R}$ is called a neighbourhood of r if:

(Def. 7)[4] There exists g such that $0 < g$ and it $=]r-g, r+g[$.

Let r be a real number. Observe that every neighbourhood of r is open.

Next we state several propositions:

(37)[5] For every neighbourhood N of r holds $r \in N$.

(38) For every r and for all neighbourhoods N_1, N_2 of r there exists a neighbourhood N of r such that $N \subseteq N_1$ and $N \subseteq N_2$.

(39) For every open subset X of $\mathbb{R}$ and for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$.

(40) For every open subset X of $\mathbb{R}$ and for every r such that $r \in X$ there exists g such that $0 < g$ and $]r-g, r+g[\subseteq X$.

(41) If for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$, then X is open.

(42) For every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ iff X is open.

(43) If X is open and upper bounded, then $\sup X \notin X$.

(44) If X is open and lower bounded, then $\inf X \notin X$.

(45) If X is open and bounded and for all g_1, g_2 such that $g_1 \in X$ and $g_2 \in X$ holds $[g_1, g_2] \subseteq X$, then there exist p, g such that $X =]p, g[$.

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[4] The definition (Def. 6) has been removed.

[5] The propositions (34)–(36) have been removed.

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