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High Speed Modulo Calculation Algorithm with Radix-2^k SD Number

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Summary. In RSA Cryptograms, many modulo calculations are used, but modulo calculation is based on many subtractions and it takes long time to calculate. In this article, we explain about a new modulo calculation algorithm using table. And we proof that upper 3 digits of Radix-2^k SD numbers is enough to specify the answer. In the first section, we prepared some useful theorems for operations of Radix-2^k SD Number. In the second section, we defined Upper 3 Digits of Radix-2^k SD number and proved that property. In the third section, we proved some property about the minimum digits of Radix-2^k SD number. In the fourth section, we identified the range of modulo arithmetic result and proved that the Upper 3 Digits indicate two possible answers. And in the last section, we defined a function to select true answer from the results of Upper 3 Digits.

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The articles [8], [10], [9], [1], [7], [4], [2], [3], [11], [5], and [6] provide the notation and terminology for this paper.

1. Some Useful Theorems

The following two propositions are true:

(1) Let n be a natural number. Suppose $n \geq 1$. Let m, k be natural numbers. If $m \geq 1$ and $k \geq 2$, then $\mathrm{SDDec\,Fmin}(m+n,m,k) = \mathrm{SDDec\,Fmin}(m,m,k)$.

(2) For all natural numbers m, k such that $m \geq 1$ and $k \geq 2$ holds $\mathrm{SDDec\,Fmin}(m,m,k) > 0$.

2. Definitions of Upper 3 Digits of Radix-2^k SD Number and Its Property

Let i, m, k be natural numbers and let r be a $m+2$-tuple of $k-\mathrm{SD}$. Let us assume that $i \in \mathrm{Seg}(m+2)$. The functor $\mathrm{M0Digit}(r,i)$ yields an element of $k-\mathrm{SD}$ and is defined as follows:

(Def. 1)(i) $\mathrm{M0Digit}(r,i) = r(i)$ if $i \geq m$,

(ii) $\mathrm{M0Digit}(r,i) = 0$ if $i < m$.

Let m, k be natural numbers and let r be a $m+2$-tuple of $k-\mathrm{SD}$. The functor $\mathrm{M0}(r)$ yielding a $m+2$-tuple of $k-\mathrm{SD}$ is defined by:

(Def. 2) For every natural number i such that $i \in \mathrm{Seg}(m+2)$ holds $\mathrm{DigA}(\mathrm{M0}(r),i) = \mathrm{M0Digit}(r,i)$.

Let i, m, k be natural numbers and let r be a $m+2$-tuple of $k-$SD. Let us assume that $k \geq 2$ and $i \in \mathrm{Seg}(m+2)$. The functor $\mathrm{MmaxDigit}(r,i)$ yielding an element of $k-$SD is defined by:

(Def. 3)(i) $\mathrm{MmaxDigit}(r,i) = r(i)$ if $i \geq m$,

(ii) $\mathrm{MmaxDigit}(r,i) = \mathrm{Radix}\,k - 1$ if $i < m$.

Let m, k be natural numbers and let r be a $m+2$-tuple of $k-$SD. The functor $\mathrm{Mmax}(r)$ yields a $m+2$-tuple of $k-$SD and is defined by:

(Def. 4) For every natural number i such that $i \in \mathrm{Seg}(m+2)$ holds $\mathrm{DigA}(\mathrm{Mmax}(r),i) = \mathrm{MmaxDigit}(r,i)$.

Let i, m, k be natural numbers and let r be a $m+2$-tuple of $k-$SD. Let us assume that $k \geq 2$ and $i \in \mathrm{Seg}(m+2)$. The functor $\mathrm{MminDigit}(r,i)$ yielding an element of $k-$SD is defined by:

(Def. 5)(i) $\mathrm{MminDigit}(r,i) = r(i)$ if $i \geq m$,

(ii) $\mathrm{MminDigit}(r,i) = -\mathrm{Radix}\,k + 1$ if $i < m$.

Let m, k be natural numbers and let r be a $m+2$-tuple of $k-$SD. The functor $\mathrm{Mmin}(r)$ yielding a $m+2$-tuple of $k-$SD is defined by:

(Def. 6) For every natural number i such that $i \in \mathrm{Seg}(m+2)$ holds $\mathrm{DigA}(\mathrm{Mmin}(r),i) = \mathrm{MminDigit}(r,i)$.

We now state two propositions:

(3) For all natural numbers m, k such that $m \geq 1$ and $k \geq 2$ and for every $m+2$-tuple r of $k-$SD holds $\mathrm{SDDec}\,\mathrm{Mmax}(r) \geq \mathrm{SDDec}\,r$.

(4) For all natural numbers m, k such that $m \geq 1$ and $k \geq 2$ and for every $m+2$-tuple r of $k-$SD holds $\mathrm{SDDec}\,r \geq \mathrm{SDDec}\,\mathrm{Mmin}(r)$.

3. Properties of Minimum Digits of Radix-2^k SD Number

Let n, k be natural numbers and let x be an integer. We say that x needs digits of n, k if and only if:

(Def. 7) $x < (\mathrm{Radix}\,k)^n$ and $x \geq (\mathrm{Radix}\,k)^{n-'1}$.

The following three propositions are true:

(5) For all natural numbers x, n, k, i such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{DigA}(\mathrm{DecSD}(x,n,k),i) \geq 0$.

(6) For all natural numbers n, k, x such that $n \geq 1$ and $k \geq 2$ and x needs digits of n, k holds $\mathrm{DigA}(\mathrm{DecSD}(x,n,k),n) > 0$.

(7) For all natural numbers f, m, k such that $m \geq 1$ and $k \geq 2$ and f needs digits of m, k holds $f \geq \mathrm{SDDec}\,\mathrm{Fmin}(m+2,m,k)$.

4. Modulo Calculation Algorithm Using Upper 3 Digits of Radix-2^k SD Number

One can prove the following propositions:

(8) For all integers m_1, m_2, f such that $m_2 < m_1 + f$ and $f > 0$ there exists an integer s such that $-f < m_1 - s \cdot f$ and $m_2 - s \cdot f < f$.

(9) Let m, k be natural numbers. Suppose $m \geq 1$ and $k \geq 2$. Let r be a $m+2$-tuple of $k-$SD. Then $\mathrm{SDDec}\,\mathrm{Mmax}(r) + \mathrm{SDDec}\,\mathrm{DecSD}(0,m+2,k) = \mathrm{SDDec}\,\mathrm{M0}(r) + \mathrm{SDDec}\,\mathrm{SDMax}(m+2,m,k)$.

(10) For all natural numbers m, k such that $m \geq 1$ and $k \geq 2$ and for every $m+2$-tuple r of $k-\mathrm{SD}$ holds $\mathrm{SDDec}\,\mathrm{Mmax}(r) < \mathrm{SDDec}\,\mathrm{M0}(r) + \mathrm{SDDec}\,\mathrm{Fmin}(m+2,m,k)$.

(11) Let m, k be natural numbers. Suppose $m \geq 1$ and $k \geq 2$. Let r be a $m+2$-tuple of $k-\mathrm{SD}$. Then $\mathrm{SDDec}\,\mathrm{Mmin}(r) + \mathrm{SDDec}\,\mathrm{DecSD}(0,m+2,k) = \mathrm{SDDec}\,\mathrm{M0}(r) + \mathrm{SDDec}\,\mathrm{SDMin}(m+2,m,k)$.

(12) Let m, k be natural numbers and r be a $m+2$-tuple of $k-\mathrm{SD}$. If $m \geq 1$ and $k \geq 2$, then $\mathrm{SDDec}\,\mathrm{M0}(r)+\mathrm{SDDec}\,\mathrm{DecSD}(0,m+2,k) = \mathrm{SDDec}\,\mathrm{Mmin}(r)+\mathrm{SDDec}\,\mathrm{SDMax}(m+2,m,k)$.

(13) For all natural numbers m, k such that $m \geq 1$ and $k \geq 2$ and for every $m+2$-tuple r of $k-\mathrm{SD}$ holds $\mathrm{SDDec}\,\mathrm{M0}(r) < \mathrm{SDDec}\,\mathrm{Mmin}(r) + \mathrm{SDDec}\,\mathrm{Fmin}(m+2,m,k)$.

(14) Let m, k, f be natural numbers and r be a $m+2$-tuple of $k-\mathrm{SD}$. Suppose $m \geq 1$ and $k \geq 2$ and f needs digits of m, k. Then there exists an integer s such that $-f < \mathrm{SDDec}\,\mathrm{M0}(r) - s \cdot f$ and $\mathrm{SDDec}\,\mathrm{Mmax}(r) - s \cdot f < f$.

(15) Let m, k, f be natural numbers and r be a $m+2$-tuple of $k-\mathrm{SD}$. Suppose $m \geq 1$ and $k \geq 2$ and f needs digits of m, k. Then there exists an integer s such that $-f < \mathrm{SDDec}\,\mathrm{Mmin}(r) - s \cdot f$ and $\mathrm{SDDec}\,\mathrm{M0}(r) - s \cdot f < f$.

(16) Let m, k be natural numbers and r be a $m+2$-tuple of $k-\mathrm{SD}$. If $m \geq 1$ and $k \geq 2$, then $\mathrm{SDDec}\,\mathrm{M0}(r) \leq \mathrm{SDDec}\,r$ and $\mathrm{SDDec}\,r \leq \mathrm{SDDec}\,\mathrm{Mmax}(r)$ or $\mathrm{SDDec}\,\mathrm{Mmin}(r) \leq \mathrm{SDDec}\,r$ and $\mathrm{SDDec}\,r < \mathrm{SDDec}\,\mathrm{M0}(r)$.

5. How to Identify the Range of Modulo Arithmetic Result

Let i, m, k be natural numbers and let r be a $m+2$-tuple of $k-\mathrm{SD}$. Let us assume that $i \in \mathrm{Seg}(m+2)$. The functor $\mathrm{MmaskDigit}(r,i)$ yielding an element of $k-\mathrm{SD}$ is defined as follows:

(Def. 8)(i) $\mathrm{MmaskDigit}(r,i) = r(i)$ if $i < m$,

(ii) $\mathrm{MmaskDigit}(r,i) = 0$ if $i \geq m$.

Let m, k be natural numbers and let r be a $m+2$-tuple of $k-\mathrm{SD}$. The functor $\mathrm{Mmask}(r)$ yielding a $m+2$-tuple of $k-\mathrm{SD}$ is defined by:

(Def. 9) For every natural number i such that $i \in \mathrm{Seg}(m+2)$ holds $\mathrm{DigA}(\mathrm{Mmask}(r),i) = \mathrm{MmaskDigit}(r,i)$.

Next we state two propositions:

(17) For all natural numbers m, k and for every $m+2$-tuple r of $k-\mathrm{SD}$ such that $m \geq 1$ and $k \geq 2$ holds $\mathrm{SDDec}\,\mathrm{M0}(r) + \mathrm{SDDec}\,\mathrm{Mmask}(r) = \mathrm{SDDec}\,r + \mathrm{SDDec}\,\mathrm{DecSD}(0,m+2,k)$.

(18) For all natural numbers m, k and for every $m+2$-tuple r of $k-\mathrm{SD}$ such that $m \geq 1$ and $k \geq 2$ holds if $\mathrm{SDDec}\,\mathrm{Mmask}(r) > 0$, then $\mathrm{SDDec}\,r > \mathrm{SDDec}\,\mathrm{M0}(r)$.

Let i, m, k be natural numbers. Let us assume that $k \geq 2$. The functor $\mathrm{FSDMinDigit}(m,k,i)$ yields an element of $k-\mathrm{SD}$ and is defined as follows:

$$\text{(Def. 10)}\quad \mathrm{FSDMinDigit}(m,k,i) = \begin{cases} \text{(i)} \quad 0, \text{ if } i > m, \\ \text{(ii)} \quad 1, \text{ if } i = m, \\ -\mathrm{Radix}\,k+1, \text{ otherwise.} \end{cases}$$

Let n, m, k be natural numbers. The functor $\mathrm{FSDMin}(n,m,k)$ yields a n-tuple of $k-\mathrm{SD}$ and is defined as follows:

(Def. 11) For every natural number i such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{DigA}(\mathrm{FSDMin}(n,m,k),i) = \mathrm{FSDMinDigit}(m,k,i)$.

The following proposition is true

(19) For every natural number n such that $n \geq 1$ and for all natural numbers m, k such that $m \in \mathrm{Seg}\, n$ and $k \geq 2$ holds $\mathrm{SDDec}\, \mathrm{FSDMin}(n,m,k) = 1$.

Let n, m, k be natural numbers and let r be a $m+2$-tuple of $k-\mathrm{SD}$. We say that r is zero over n if and only if:

(Def. 12) For every natural number i such that $i > n$ holds $\mathrm{DigA}(r,i) = 0$.

One can prove the following proposition

(20) Let m be a natural number. Suppose $m \geq 1$. Let n, k be natural numbers and r be a $m+2$-tuple of $k-\mathrm{SD}$. If $k \geq 2$ and $n \in \mathrm{Seg}(m+2)$ and $\mathrm{Mmask}(r)$ is zero over n and $\mathrm{DigA}(\mathrm{Mmask}(r),n) > 0$, then $\mathrm{SDDec}\, \mathrm{Mmask}(r) > 0$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Yoshinori Fujisawa and Yasushi Fuwa. Definitions of radix-2^k signed-digit number and its adder algorithm. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/radix_1.html`.

[5] Andrzej Kondracki. The Chinese Remainder Theorem. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/wsierp_1.html`.

[6] Masaaki Niimura and Yasushi Fuwa. Magnitude relation properties of radix-2^k sd number. *Journal of Formalized Mathematics*, 15, 2003. `http://mizar.org/JFM/Vol15/radix_5.html`.

[7] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/binarith.html`.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[9] Michał J. Trybulec. Integers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/int_1.html`.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[11] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

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