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The Subformula Tree of a Formula of the First Order Language

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Summary. A continuation of [12]. The notions of list of immediate constituents of a formula and subformula tree of a formula are introduced. The some propositions related to these notions are proved.

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The articles [15], [11], [19], [17], [3], [20], [9], [10], [13], [8], [18], [1], [4], [5], [6], [7], [14], [2], and [16] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following propositions are true:

(4)[1] For every natural number n and for every finite sequence r there exists a finite sequence q such that $q = r \restriction \mathrm{Seg}\, n$ and $q \preceq r$.

(6)[2] Let D be a non empty set, r be a finite sequence of elements of D, r_1, r_2 be finite sequences, and k be a natural number. Suppose $k+1 \leq \mathrm{len}\, r$ and $r_1 = r \restriction \mathrm{Seg}(k+1)$ and $r_2 = r \restriction \mathrm{Seg}\, k$. Then there exists an element x of D such that $r_1 = r_2 \frown \langle x \rangle$.

(7) Let D be a non empty set, r be a finite sequence of elements of D, and r_1 be a finite sequence. If $1 \leq \mathrm{len}\, r$ and $r_1 = r \restriction \mathrm{Seg}\, 1$, then there exists an element x of D such that $r_1 = \langle x \rangle$.

Let D be a non empty set and let T be a tree decorated with elements of D. Observe that every element of $\mathrm{dom}\, T$ is function-like and relation-like.

Let D be a non empty set and let T be a tree decorated with elements of D. Note that every element of $\mathrm{dom}\, T$ is finite sequence-like.

Let D be a non empty set. Observe that there exists a tree decorated with elements of D which is finite.

In the sequel T denotes a decorated tree and p denotes a finite sequence of elements of $\mathbb{N}$.

One can prove the following proposition

(8) $T(p) = (T \restriction p)(\emptyset)$.

[1] The propositions (1)–(3) have been removed.
[2] The proposition (5) has been removed.

In the sequel T is a finite-branching decorated tree, t is an element of $\operatorname{dom} T$, x is a finite sequence, and n is a natural number.

Next we state several propositions:

(9) $\operatorname{succ}(T,t) = T \cdot \operatorname{Succ} t$.

(10) $\operatorname{dom}(T \cdot \operatorname{Succ} t) = \operatorname{dom} \operatorname{Succ} t$.

(11) $\operatorname{dom} \operatorname{succ}(T,t) = \operatorname{dom} \operatorname{Succ} t$.

(12) $t \frown \langle n \rangle \in \operatorname{dom} T$ iff $n+1 \in \operatorname{dom} \operatorname{Succ} t$.

(13) For all T, x, n such that $x \frown \langle n \rangle \in \operatorname{dom} T$ holds $T(x \frown \langle n \rangle) = (\operatorname{succ}(T,x))(n+1)$.

In the sequel x, x' denote elements of $\operatorname{dom} T$ and y' denotes a set.

Next we state two propositions:

(14) If $x' \in \operatorname{succ} x$, then $T(x') \in \operatorname{rng} \operatorname{succ}(T,x)$.

(15) If $y' \in \operatorname{rng} \operatorname{succ}(T,x)$, then there exists x' such that $y' = T(x')$ and $x' \in \operatorname{succ} x$.

In the sequel n, k, m denote natural numbers.

The scheme *ExDecTrees* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a finite sequence of elements of $\mathcal{A}$, and states that:

There exists a finite-branching tree T decorated with elements of $\mathcal{A}$ such that $T(\emptyset) = \mathcal{B}$ and for every element t of $\operatorname{dom} T$ and for every element w of $\mathcal{A}$ such that $w = T(t)$ holds $\operatorname{succ}(T,t) = \mathcal{F}(w)$

for all values of the parameters.

We now state a number of propositions:

(16) For every tree T and for every element t of T holds $\operatorname{Seg}_{\preceq}(t)$ is a finite chain of T.

(17) For every tree T holds T-level$(0) = \{\emptyset\}$.

(18) For every tree T holds T-level$(n+1) = \bigcup\{\operatorname{succ} w; w$ ranges over elements of T: $\operatorname{len} w = n\}$.

(19) For every finite-branching tree T and for every natural number n holds T-level(n) is finite.

(20) For every finite-branching tree T holds T is finite iff there exists a natural number n such that T-level$(n) = \emptyset$.

(21) For every finite-branching tree T such that T is not finite there exists a chain C of T such that C is not finite.

(22) For every finite-branching tree T such that T is not finite there exists a branch B of T such that B is not finite.

(23) Let T be a tree, C be a chain of T, and t be an element of T. If $t \in C$ and C is not finite, then there exists an element t' of T such that $t' \in C$ and $t \prec t'$.

(24) Let T be a tree, B be a branch of T, and t be an element of T. Suppose $t \in B$ and B is not finite. Then there exists an element t' of T such that $t' \in B$ and $t' \in \operatorname{succ} t$.

(25) Let f be a function from $\mathbb{N}$ into $\mathbb{N}$. Suppose that for every n holds $f(n+1)$ **qua** natural number $\leq f(n)$ **qua** natural number. Then there exists m such that for every n such that $m \leq n$ holds $f(n) = f(m)$.

The scheme *FinDecTree* deals with a non empty set $\mathcal{A}$, a finite-branching tree $\mathcal{B}$ decorated with elements of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a natural number, and states that:

$\mathcal{B}$ is finite

provided the parameters meet the following condition:

- For all elements t, t' of $\operatorname{dom} \mathcal{B}$ and for every element d of $\mathcal{A}$ such that $t' \in \operatorname{succ} t$ and $d = \mathcal{B}(t')$ holds $\mathcal{F}(d) < \mathcal{F}(\mathcal{B}(t))$.

In the sequel D denotes a non empty set and T denotes a tree decorated with elements of D.

One can prove the following two propositions:

(26) For every set y such that $y \in \operatorname{rng} T$ holds y is an element of D.

(27) For every set x such that $x \in \operatorname{dom} T$ holds $T(x)$ is an element of D.

2. Subformula tree

In the sequel F, G, H denote elements of WFF.

The following two propositions are true:

(28) If F is a subformula of G, then $\operatorname{len}({}^{@}F) \leq \operatorname{len}({}^{@}G)$.

(29) If F is a subformula of G and $\operatorname{len}({}^{@}F) = \operatorname{len}({}^{@}G)$, then $F = G$.

Let p be an element of WFF. The list of immediate constituents of p yielding a finite sequence of elements of WFF is defined by:

(Def. 1) $$\text{The list of immediate constituents of } p = \begin{cases} \varepsilon_{\mathrm{WFF}}, \text{ if } p = \mathrm{VERUM} \text{ or } p \text{is atomic}, \\ \langle \mathrm{Arg}(p) \rangle, \text{ if } p \text{is negative}, \\ \langle \mathrm{LeftArg}(p), \mathrm{RightArg}(p) \rangle, \text{ if } p \text{is conjunctive}, \\ \langle \mathrm{Scope}(p) \rangle, \text{ otherwise.} \end{cases}$$

We now state two propositions:

(30) Suppose $k \in \operatorname{dom}$(the list of immediate constituents of F) and $G =$ (the list of immediate constituents of F)(k). Then G is an immediate constituent of F.

(31) rng(the list of immediate constituents of F) $= \{G; G$ ranges over elements of WFF: G is an immediate constituent of $F\}$.

Let p be an element of WFF. The tree of subformulae of p yields a finite tree decorated with elements of WFF and is defined by the conditions (Def. 2).

(Def. 2)(i) (The tree of subformulae of p)$(\emptyset) = p$, and

(ii) for every element x of dom(the tree of subformulae of p) holds succ(the tree of subformulae of p, x) $=$ the list of immediate constituents of (the tree of subformulae of p)(x).

In the sequel t, t' denote elements of dom(the tree of subformulae of F).

Next we state a number of propositions:

(34)[3] $F \in \operatorname{rng}$(the tree of subformulae of F).

(35) Suppose $t \mathbin{^\frown} \langle n \rangle \in \operatorname{dom}$(the tree of subformulae of F). Then there exists G such that

(i) $G =$ (the tree of subformulae of F)$(t \mathbin{^\frown} \langle n \rangle)$, and

(ii) G is an immediate constituent of (the tree of subformulae of F)(t).

(36) The following statements are equivalent

(i) H is an immediate constituent of (the tree of subformulae of F)(t),

(ii) there exists n such that $t \mathbin{^\frown} \langle n \rangle \in \operatorname{dom}$(the tree of subformulae of F) and $H =$ (the tree of subformulae of F)$(t \mathbin{^\frown} \langle n \rangle)$.

(37) Suppose $G \in \operatorname{rng}$(the tree of subformulae of F) and H is an immediate constituent of G. Then $H \in \operatorname{rng}$(the tree of subformulae of F).

(38) If $G \in \operatorname{rng}$(the tree of subformulae of F) and H is a subformula of G, then $H \in \operatorname{rng}$(the tree of subformulae of F).

(39) $G \in \operatorname{rng}$(the tree of subformulae of F) iff G is a subformula of F.

[3] The propositions (32) and (33) have been removed.

(40) rng(the tree of subformulae of F) $=$ Subformulae$F$.

(41) Suppose $t' \in \operatorname{succ} t$. Then (the tree of subformulae of $F)(t')$ is an immediate constituent of (the tree of subformulae of $F)(t)$.

(42) If $t \preceq t'$, then (the tree of subformulae of $F)(t')$ is a subformula of (the tree of subformulae of $F)(t)$.

(43) If $t \prec t'$, then len($^{@}$(the tree of subformulae of $F)(t')) <$ len($^{@}$(the tree of subformulae of $F)(t)$).

(44) If $t \prec t'$, then (the tree of subformulae of $F)(t') \neq$ (the tree of subformulae of $F)(t)$.

(45) If $t \prec t'$, then (the tree of subformulae of $F)(t')$ is a proper subformula of (the tree of subformulae of $F)(t)$.

(46) (The tree of subformulae of $F)(t) = F$ iff $t = \emptyset$.

(47) Suppose $t \neq t'$ and (the tree of subformulae of $F)(t) =$ (the tree of subformulae of $F)(t')$. Then t and t' are not $\subseteq$-comparable.

Let F, G be elements of WFF. The F-entry points in subformula tree of G yields an antichain of prefixes of dom(the tree of subformulae of F) and is defined by the condition (Def. 3).

(Def. 3) Let t be an element of dom(the tree of subformulae of F). Then $t \in$ the F-entry points in subformula tree of G if and only if (the tree of subformulae of $F)(t) = G$.

Next we state several propositions:

(49)[4] The F-entry points in subformula tree of $G = \{t; t$ ranges over elements of dom(the tree of subformulae of F): (the tree of subformulae of $F)(t) = G\}$.

(50) G is a subformula of F iff the F-entry points in subformula tree of $G \neq \emptyset$.

(51) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of $F)(t)$ is negative. Then (the tree of subformulae of $F)(t') =$ Arg((the tree of subformulae of $F)(t)$) and $m = 0$.

(52) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of $F)(t)$ is conjunctive. Then

(i) (the tree of subformulae of $F)(t') =$ LeftArg((the tree of subformulae of $F)(t)$) and $m = 0$, or

(ii) (the tree of subformulae of $F)(t') =$ RightArg((the tree of subformulae of $F)(t)$) and $m = 1$.

(53) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of $F)(t)$ is universal. Then (the tree of subformulae of $F)(t') =$ Scope((the tree of subformulae of $F)(t)$) and $m = 0$.

(54) Suppose (the tree of subformulae of $F)(t)$ is negative. Then

(i) $t \frown \langle 0 \rangle \in$ dom(the tree of subformulae of F), and

(ii) (the tree of subformulae of $F)(t \frown \langle 0 \rangle) =$ Arg((the tree of subformulae of $F)(t)$).

(55) Suppose (the tree of subformulae of $F)(t)$ is conjunctive. Then

(i) $t \frown \langle 0 \rangle \in$ dom(the tree of subformulae of F),

(ii) (the tree of subformulae of $F)(t \frown \langle 0 \rangle) =$ LeftArg((the tree of subformulae of $F)(t)$),

(iii) $t \frown \langle 1 \rangle \in$ dom(the tree of subformulae of F), and

(iv) (the tree of subformulae of $F)(t \frown \langle 1 \rangle) =$ RightArg((the tree of subformulae of $F)(t)$).

[4] The proposition (48) has been removed.

(56) Suppose (the tree of subformulae of $F)(t)$ is universal. Then

(i) $t \frown \langle 0 \rangle \in$ dom(the tree of subformulae of F), and

(ii) (the tree of subformulae of $F)(t \frown \langle 0 \rangle) =$ Scope((the tree of subformulae of $F)(t))$.

In the sequel t denotes an element of dom(the tree of subformulae of F) and s denotes an element of dom(the tree of subformulae of G).

Next we state the proposition

(57) Suppose $t \in$ the F-entry points in subformula tree of G and $s \in$ the G-entry points in subformula tree of H. Then $t \frown s \in$ the F-entry points in subformula tree of H.

In the sequel t denotes an element of dom(the tree of subformulae of F) and s denotes a finite sequence.

Next we state several propositions:

(58) Suppose $t \in$ the F-entry points in subformula tree of G and $t \frown s \in$ the F-entry points in subformula tree of H. Then $s \in$ the G-entry points in subformula tree of H.

(59) Let given F, G, H. Then $\{t \frown s; t$ ranges over elements of dom(the tree of subformulae of F), s ranges over elements of dom(the tree of subformulae of G): $t \in$ the F-entry points in subformula tree of $G \land s \in$ the G-entry points in subformula tree of $H\} \subseteq$ the F-entry points in subformula tree of H.

(60) (The tree of subformulae of $F) \restriction t =$ the tree of subformulae of (the tree of subformulae of $F)(t)$.

(61) $t \in$ the F-entry points in subformula tree of G if and only if (the tree of subformulae of $F) \restriction t =$ the tree of subformulae of G.

(62) The F-entry points in subformula tree of $G = \{t; t$ ranges over elements of dom(the tree of subformulae of F): (the tree of subformulae of $F) \restriction t =$ the tree of subformulae of $G\}$.

In the sequel C is a chain of dom(the tree of subformulae of F).

One can prove the following proposition

(63) Let given F, G, H, C. Suppose that

(i) $G \in \{$(the tree of subformulae of $F)(t); t$ ranges over elements of dom(the tree of subformulae of F): $t \in C\}$, and

(ii) $H \in \{$(the tree of subformulae of $F)(t); t$ ranges over elements of dom(the tree of subformulae of F): $t \in C\}$.

Then G is a subformula of H or H is a subformula of G.

Let F be an element of WFF. An element of WFF is called a subformula of F if:

(Def. 4) It is a subformula of F.

Let F be an element of WFF and let G be a subformula of F. An element of dom(the tree of subformulae of F) is said to be an entry point in subformula tree of G if:

(Def. 5) (The tree of subformulae of $F)(\mathrm{it}) = G$.

In the sequel G denotes a subformula of F and t, t' denote entry points in subformula tree of G.

Next we state the proposition

(65)[5] If $t \neq t'$, then t and t' are not $\subseteq$-comparable.

[5] The proposition (64) has been removed.

Let F be an element of WFF and let G be a subformula of F. The entry points in subformula tree of G yields a non empty antichain of prefixes of dom(the tree of subformulae of F) and is defined by:

(Def. 6) The entry points in subformula tree of $G =$ the F-entry points in subformula tree of G.

We now state two propositions:

(67)[6] $t \in$ the entry points in subformula tree of G.

(68) The entry points in subformula tree of $G = \{t; t$ ranges over entry points in subformula tree of G: $t = t\}$.

In the sequel G_1, G_2 denote subformulae of F, t_1 denotes an entry point in subformula tree of G_1, and s denotes an element of dom(the tree of subformulae of G_1).

Next we state the proposition

(69) If $s \in$ the G_1-entry points in subformula tree of G_2, then $t_1 \frown s$ is an entry point in subformula tree of G_2.

In the sequel s is a finite sequence.

We now state three propositions:

(70) If $t_1 \frown s$ is an entry point in subformula tree of G_2, then $s \in$ the G_1-entry points in subformula tree of G_2.

(71) Let given F, G_1, G_2. Then $\{t \frown s; t$ ranges over entry points in subformula tree of G_1, s ranges over elements of dom(the tree of subformulae of G_1): $s \in$ the G_1-entry points in subformula tree of $G_2\} = \{t \frown s; t$ ranges over elements of dom(the tree of subformulae of F), s ranges over elements of dom(the tree of subformulae of G_1): $t \in$ the F-entry points in subformula tree of $G_1 \ \wedge\ s \in$ the G_1-entry points in subformula tree of $G_2\}$.

(72) Let given F, G_1, G_2. Then $\{t \frown s; t$ ranges over entry points in subformula tree of G_1, s ranges over elements of dom(the tree of subformulae of G_1): $s \in$ the G_1-entry points in subformula tree of $G_2\} \subseteq$ the entry points in subformula tree of G_2.

In the sequel G_1, G_2 are subformulae of F, t_1 is an entry point in subformula tree of G_1, and t_2 is an entry point in subformula tree of G_2.

Next we state two propositions:

(73) If there exist t_1, t_2 such that $t_1 \preceq t_2$, then G_2 is a subformula of G_1.

(74) If G_2 is a subformula of G_1, then for every t_1 there exists t_2 such that $t_1 \preceq t_2$.

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