JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

Minimal Signature for Partial Algebra

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Summary. The concept of characterizing of partial algebras by many sorted signature is introduced, i.e. we say that a signature S characterizes a partial algebra A if there is an S-algebra whose sorts form a partition of the carrier of algebra A and operations are formed from operations of A by the partition. The main result is that for any partial algebra there is the minimal many sorted signature which characterizes the algebra. The minimality means that there are signature endomorphisms from any signature which characterizes the algebra A onto the minimal one.

MML Identifier: `PUA2MSS1`.

WWW: `http://mizar.org/JFM/Vol7/pua2mss1.html`

The articles [13], [7], [18], [17], [1], [11], [19], [20], [4], [14], [3], [8], [6], [5], [21], [12], [10], [2], [15], [16], and [9] provide the notation and terminology for this paper.

1. Preliminary

Let f be a non-empty function. Note that $\operatorname{rng} f$ has non empty elements.

Let X, Y be non empty sets. One can check that there exists a partial function from X to Y which is non empty.

Let X be a set with non empty elements. One can verify that every finite sequence of elements of X is non-empty.

Let A be a non empty set. One can verify that there exists a finite sequence of operational functions of A which is homogeneous, quasi total, non-empty, and non empty.

Let us note that every universal algebra structure which is non-empty is also non empty.

Let X be a non empty set with non empty elements. Observe that every element of X is non empty.

The following propositions are true:

(1) For all non-empty functions f, g such that $\prod f \subseteq \prod g$ holds $\operatorname{dom} f = \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \subseteq g(x)$.

(2) For all non-empty functions f, g such that $\prod f = \prod g$ holds $f = g$.

Let A be a non empty set and let f be a finite sequence of operational functions of A. Then $\operatorname{rng} f$ is a subset of $A^* \dot{\to} A$.

Let A, B be non empty sets and let S be a non empty subset of $A \dot{\to} B$. We see that the element of S is a partial function from A to B.

Let A be a non-empty universal algebra structure. An operation symbol of A is an element of dom(the characteristic of A). An operation of A is an element of rng(the characteristic of A).

Let A be a non-empty universal algebra structure and let o be an operation symbol of A. The functor $\operatorname{Den}(o, A)$ yields an operation of A and is defined by:

(Def. 1) $\mathrm{Den}(o,A) = (\text{the characteristic of } A)(o)$.

2. Partitions

Let X be a set. Note that every partition of X has non empty elements.

Let X be a set and let R be an equivalence relation of X. Then $\mathrm{Classes}\,R$ is a partition of X.

Next we state a number of propositions:

(3) Let X be a set, P be a partition of X, and x, a, b be sets. If $x \in a$ and $a \in P$ and $x \in b$ and $b \in P$, then $a = b$.

(4) Let X, Y be sets. Suppose X is finer than Y. Let p be a finite sequence of elements of X. Then there exists a finite sequence q of elements of Y such that $\prod p \subseteq \prod q$.

(5) Let X be a set, P, Q be partitions of X, and f be a function from P into Q. Suppose that for every set a such that $a \in P$ holds $a \subseteq f(a)$. Let p be a finite sequence of elements of P and q be a finite sequence of elements of Q. Then $\prod p \subseteq \prod q$ if and only if $f \cdot p = q$.

(6) For every set P and for every function f such that $\mathrm{rng}\, f \subseteq \bigcup P$ there exists a function p such that $\mathrm{dom}\, p = \mathrm{dom}\, f$ and $\mathrm{rng}\, p \subseteq P$ and $f \in \prod p$.

(7) Let X be a set, P be a partition of X, and f be a finite sequence of elements of X. Then there exists a finite sequence p of elements of P such that $f \in \prod p$.

(8) Let X, Y be non empty sets, P be a partition of X, and Q be a partition of Y. Then $\{[\!:p, q:\!] : p$ ranges over elements of P, q ranges over elements of $Q\}$ is a partition of $[\!:X, Y:\!]$.

(9) For every non empty set X and for every partition P of X holds $\{\prod p : p$ ranges over elements of $P^*\}$ is a partition of X^*.

(10) Let X be a non empty set, n be a natural number, and P be a partition of X. Then $\{\prod p : p$ ranges over elements of $P^n\}$ is a partition of X^n.

(11) Let X be a non empty set and Y be a set. Suppose $Y \subseteq X$. Let P be a partition of X. Then $\{a \cap Y; a$ ranges over elements of P: a meets $Y\}$ is a partition of Y.

(12) Let f be a non empty function and P be a partition of $\mathrm{dom}\, f$. Then $\{f{\restriction}a : a$ ranges over elements of $P\}$ is a partition of f.

Let X be a set. The functor $\mathrm{SmallestPartition}(X)$ yielding a partition of X is defined as follows:

(Def. 2) $\mathrm{SmallestPartition}(X) = \mathrm{Classes}(\mathrm{id}_X)$.

One can prove the following propositions:

(13) For every non empty set X holds $\mathrm{SmallestPartition}(X) = \{\{x\} : x$ ranges over elements of $X\}$.

(14) Let X be a set and p be a finite sequence of elements of $\mathrm{SmallestPartition}(X)$. Then there exists a finite sequence q of elements of X such that $\prod p = \{q\}$.

Let X be a set. A function is called an indexed partition of X if:

(Def. 3) $\mathrm{rng}\,\mathrm{it}$ is a partition of X and it is one-to-one.

Let X be a set. Note that every indexed partition of X is one-to-one and non-empty. Let P be an indexed partition of X. Then $\mathrm{rng}\, P$ is a partition of X.

Let X be a non empty set. Note that every indexed partition of X is non empty.

Let X be a set and let P be a partition of X. Then id_P is an indexed partition of X.

Let X be a set, let P be an indexed partition of X, and let x be a set. Let us assume that $x \in X$. The P-index of x is a set and is defined by:

(Def. 4) The P-index of $x \in \operatorname{dom} P$ and $x \in P(\text{the } P\text{-index of } x)$.

The following propositions are true:

(15) Let X be a set and P be a non-empty function. Suppose $\bigcup P = X$ and for all sets x, y such that $x \in \operatorname{dom} P$ and $y \in \operatorname{dom} P$ and $x \neq y$ holds $P(x)$ misses $P(y)$. Then P is an indexed partition of X.

(16) Let X, Y be non empty sets, P be a partition of Y, and f be a function from X into P. If $P \subseteq \operatorname{rng} f$ and f is one-to-one, then f is an indexed partition of Y.

3. Relations Generated by Operations of Partial Algebra

In this article we present several logical schemes. The scheme *IndRelationEx* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a natural number $\mathcal{C}$, a relation $\mathcal{D}$ between $\mathcal{A}$ and $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a relation between $\mathcal{A}$ and $\mathcal{B}$, and states that:

There exists a relation R between $\mathcal{A}$ and $\mathcal{B}$ and there exists a many sorted set F indexed by $\mathbb{N}$ such that

(i) $R = F(\mathcal{C})$,

(ii) $F(0) = \mathcal{D}$, and

(iii) for every natural number i and for every relation R between $\mathcal{A}$ and $\mathcal{B}$ such that $R = F(i)$ holds $F(i+1) = \mathcal{F}(R,i)$

for all values of the parameters.

The scheme *RelationUniq* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

Let R_1, R_2 be relations between $\mathcal{A}$ and $\mathcal{B}$. Suppose that

(i) for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in R_1$ iff $\mathcal{P}[x,y]$, and

(ii) for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in R_2$ iff $\mathcal{P}[x,y]$.

Then $R_1 = R_2$

for all values of the parameters.

The scheme *IndRelationUniq* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a natural number $\mathcal{C}$, a relation $\mathcal{D}$ between $\mathcal{A}$ and $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a relation between $\mathcal{A}$ and $\mathcal{B}$, and states that:

Let R_1, R_2 be relations between $\mathcal{A}$ and $\mathcal{B}$. Suppose that

(i) there exists a many sorted set F indexed by $\mathbb{N}$ such that $R_1 = F(\mathcal{C})$ and $F(0) = \mathcal{D}$ and for every natural number i and for every relation R between $\mathcal{A}$ and $\mathcal{B}$ such that $R = F(i)$ holds $F(i+1) = \mathcal{F}(R,i)$, and

(ii) there exists a many sorted set F indexed by $\mathbb{N}$ such that $R_2 = F(\mathcal{C})$ and $F(0) = \mathcal{D}$ and for every natural number i and for every relation R between $\mathcal{A}$ and $\mathcal{B}$ such that $R = F(i)$ holds $F(i+1) = \mathcal{F}(R,i)$.

Then $R_1 = R_2$

for all values of the parameters.

Let A be a partial non-empty universal algebra structure. The functor $\operatorname{DomRel}(A)$ yielding a binary relation on the carrier of A is defined by the condition (Def. 5).

(Def. 5) Let x, y be elements of A. Then $\langle x, y\rangle \in \operatorname{DomRel}(A)$ if and only if for every operation f of A and for all finite sequences p, q holds $p \frown \langle x\rangle \frown q \in \operatorname{dom} f$ iff $p \frown \langle y\rangle \frown q \in \operatorname{dom} f$.

Let A be a partial non-empty universal algebra structure. Note that $\operatorname{DomRel}(A)$ is total, symmetric, and transitive.

Let A be a non-empty partial universal algebra structure and let R be a binary relation on the carrier of A. The functor R^A yields a binary relation on the carrier of A and is defined by the condition (Def. 6).

(Def. 6) Let x, y be elements of A. Then $\langle x, y\rangle \in R^A$ if and only if the following conditions are satisfied:

(i) $\langle x, y \rangle \in R$, and

(ii) for every operation f of A and for all finite sequences p, q such that $p \frown \langle x \rangle \frown q \in \operatorname{dom} f$ and $p \frown \langle y \rangle \frown q \in \operatorname{dom} f$ holds $\langle f(p \frown \langle x \rangle \frown q), f(p \frown \langle y \rangle \frown q) \rangle \in R$.

Let A be a non-empty partial universal algebra structure, let R be a binary relation on the carrier of A, and let i be a natural number. The functor $R^{A,i}$ yields a binary relation on the carrier of A and is defined by the condition (Def. 7).

(Def. 7) There exists a many sorted set F indexed by $\mathbb{N}$ such that

(i) $R^{A,i} = F(i)$,

(ii) $F(0) = R$, and

(iii) for every natural number i and for every binary relation R on the carrier of A such that $R = F(i)$ holds $F(i+1) = R^A$.

We now state several propositions:

(17) Let A be a non-empty partial universal algebra structure and R be a binary relation on the carrier of A. Then $R^{A,0} = R$ and $R^{A,1} = R^A$.

(18) Let A be a non-empty partial universal algebra structure, i be a natural number, and R be a binary relation on the carrier of A. Then $R^{A,i+1} = (R^{A,i})^A$.

(19) Let A be a non-empty partial universal algebra structure, i, j be natural numbers, and R be a binary relation on the carrier of A. Then $R^{A,i+j} = (R^{A,i})^{A,j}$.

(20) Let A be a non-empty partial universal algebra structure and R be an equivalence relation of the carrier of A. If $R \subseteq \mathrm{DomRel}(A)$, then R^A is total, symmetric, and transitive.

(21) Let A be a non-empty partial universal algebra structure and R be a binary relation on the carrier of A. Then $R^A \subseteq R$.

(22) Let A be a non-empty partial universal algebra structure and R be an equivalence relation of the carrier of A. Suppose $R \subseteq \mathrm{DomRel}(A)$. Let i be a natural number. Then $R^{A,i}$ is total, symmetric, and transitive.

Let A be a non-empty partial universal algebra structure. The functor $\mathrm{LimDomRel}(A)$ yielding a binary relation on the carrier of A is defined by:

(Def. 8) For all elements x, y of A holds $\langle x, y \rangle \in \mathrm{LimDomRel}(A)$ iff for every natural number i holds $\langle x, y \rangle \in (\mathrm{DomRel}(A))^{A,i}$.

The following proposition is true

(23) For every non-empty partial universal algebra structure A holds $\mathrm{LimDomRel}(A) \subseteq \mathrm{DomRel}(A)$.

Let A be a non-empty partial universal algebra structure. One can check that $\mathrm{LimDomRel}(A)$ is total, symmetric, and transitive.

4. Partitability

Let X be a non empty set, let f be a partial function from X^* to X, and let P be a partition of X. We say that f is partitable w.r.t. P if and only if:

(Def. 9) For every finite sequence p of elements of P there exists an element a of P such that $f^{\circ} \prod p \subseteq a$.

Let X be a non empty set, let f be a partial function from X^* to X, and let P be a partition of X. We say that f is exactly partitable w.r.t. P if and only if:

(Def. 10) f is partitable w.r.t. P and for every finite sequence p of elements of P such that $\prod p$ meets $\operatorname{dom} f$ holds $\prod p \subseteq \operatorname{dom} f$.

We now state the proposition

(24) Let A be a non-empty partial universal algebra structure. Then every operation of A is exactly partitable w.r.t. SmallestPartition(the carrier of A).

The scheme *FiniteTransitivity* deals with finite sequences $\mathcal{A}$, $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following requirements are met:

- $\mathcal{P}[\mathcal{A}]$,
- $\operatorname{len} \mathcal{A} = \operatorname{len} \mathcal{B}$,
- For all finite sequences p, q and for all sets z_1, z_2 such that $\mathcal{P}[p \frown \langle z_1 \rangle \frown q]$ and $\mathcal{Q}[z_1, z_2]$ holds $\mathcal{P}[p \frown \langle z_2 \rangle \frown q]$, and
- For every natural number i such that $i \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{Q}[\mathcal{A}(i), \mathcal{B}(i)]$.

The following proposition is true

(25) For every non-empty partial universal algebra structure A holds every operation of A is exactly partitable w.r.t. Classes LimDomRel(A).

Let A be a partial non-empty universal algebra structure. A partition of the carrier of A is said to be a partition of A if:

(Def. 11) Every operation of A is exactly partitable w.r.t. it.

Let A be a partial non-empty universal algebra structure. An indexed partition of the carrier of A is said to be an indexed partition of A if:

(Def. 12) rng it is a partition of A.

Let A be a partial non-empty universal algebra structure and let P be an indexed partition of A. Then $\operatorname{rng} P$ is a partition of A.

Next we state three propositions:

(26) For every non-empty partial universal algebra structure A holds Classes LimDomRel(A) is a partition of A.

(27) Let X be a non empty set, P be a partition of X, p be a finite sequence of elements of P, q_1, q_2 be finite sequences, and x, y be sets. Suppose $q_1 \frown \langle x \rangle \frown q_2 \in \prod p$ and there exists an element a of P such that $x \in a$ and $y \in a$. Then $q_1 \frown \langle y \rangle \frown q_2 \in \prod p$.

(28) For every partial non-empty universal algebra structure A holds every partition of A is finer than Classes LimDomRel(A).

5. Signature Morphisms

Let S_1, S_2 be many sorted signatures and let f, g be functions. We say that f and g form morphism between S_1 and S_2 if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) $\operatorname{dom} f =$ the carrier of S_1,

(ii) $\operatorname{dom} g =$ the operation symbols of S_1,

(iii) $\operatorname{rng} f \subseteq$ the carrier of S_2,

(iv) $\operatorname{rng} g \subseteq$ the operation symbols of S_2,

(v) $f \cdot$ the result sort of $S_1 =$ (the result sort of $S_2) \cdot g$, and

(vi) for every set o and for every function p such that $o \in$ the operation symbols of S_1 and $p =$ (the arity of $S_1)(o)$ holds $f \cdot p =$ (the arity of $S_2)(g(o))$.

One can prove the following propositions:

(29) Let S be a non void non empty many sorted signature. Then $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$ form morphism between S and S.

(30) Let S_1, S_2, S_3 be many sorted signatures and f_1, f_2, g_1, g_2 be functions. Suppose f_1 and g_1 form morphism between S_1 and S_2 and f_2 and g_2 form morphism between S_2 and S_3. Then $f_2 \cdot f_1$ and $g_2 \cdot g_1$ form morphism between S_1 and S_3.

Let S_1, S_2 be many sorted signatures. We say that S_1 is rougher than S_2 if and only if the condition (Def. 14) is satisfied.

(Def. 14) There exist functions f, g such that f and g form morphism between S_2 and S_1 and $\mathrm{rng}\, f =$ the carrier of S_1 and $\mathrm{rng}\, g =$ the operation symbols of S_1.

Let S_1, S_2 be non void non empty many sorted signatures. Let us note that the predicate S_1 is rougher than S_2 is reflexive.

Next we state the proposition

(31) For all many sorted signatures S_1, S_2, S_3 such that S_1 is rougher than S_2 and S_2 is rougher than S_3 holds S_1 is rougher than S_3.

6. Many Sorted Signature of Partial Algebra

Let A be a partial non-empty universal algebra structure and let P be a partition of A. The functor $\mathrm{MSSign}(A,P)$ yields a strict many sorted signature and is defined by the conditions (Def. 15).

(Def. 15)(i) The carrier of $\mathrm{MSSign}(A,P) = P$,

(ii) the operation symbols of $\mathrm{MSSign}(A,P) = \{\langle o, p\rangle; o$ ranges over operation symbols of A, p ranges over elements of P^*: $\prod p$ meets $\mathrm{dom}\,\mathrm{Den}(o,A)\}$, and

(iii) for every operation symbol o of A and for every element p of P^* such that $\prod p$ meets $\mathrm{dom}\,\mathrm{Den}(o,A)$ holds (the arity of $\mathrm{MSSign}(A,P))(\langle o, p\rangle) = p$ and $(\mathrm{Den}(o,A))^\circ \prod p \subseteq$ (the result sort of $\mathrm{MSSign}(A,P))(\langle o, p\rangle)$.

Let A be a partial non-empty universal algebra structure and let P be a partition of A. Note that $\mathrm{MSSign}(A,P)$ is non empty and non void.

Let A be a partial non-empty universal algebra structure, let P be a partition of A, and let o be an operation symbol of $\mathrm{MSSign}(A,P)$. Then $o_{\mathbf{1}}$ is an operation symbol of A. Then $o_{\mathbf{2}}$ is an element of P^*.

Let A be a partial non-empty universal algebra structure, let S be a non void non empty many sorted signature, let G be an algebra over S, and let P be an indexed partition of the operation symbols of S. We say that A can be characterized by S, G, and P if and only if the conditions (Def. 16) are satisfied.

(Def. 16)(i) The sorts of G are an indexed partition of A,

(ii) $\mathrm{dom}\, P = \mathrm{dom}$ (the characteristic of A), and

(iii) for every operation symbol o of A holds (the characteristics of $G) \restriction P(o)$ is an indexed partition of $\mathrm{Den}(o,A)$.

Let A be a partial non-empty universal algebra structure and let S be a non void non empty many sorted signature. We say that A can be characterized by S if and only if the condition (Def. 17) is satisfied.

(Def. 17) There exists an algebra G over S and there exists an indexed partition P of the operation symbols of S such that A can be characterized by S, G, and P.

One can prove the following propositions:

(32) Let A be a partial non-empty universal algebra structure and P be a partition of A. Then A can be characterized by $\mathrm{MSSign}(A,P)$.

(33) Let A be a partial non-empty universal algebra structure, S be a non void non empty many sorted signature, G be an algebra over S, and Q be an indexed partition of the operation symbols of S. Suppose A can be characterized by S, G, and Q. Let o be an operation symbol of A and r be a finite sequence of elements of rng(the sorts of G). Suppose $\prod r \subseteq \mathrm{dom}\,\mathrm{Den}(o, A)$. Then there exists an operation symbol s of S such that (the sorts of G) $\cdot$ $\mathrm{Arity}(s) = r$ and $s \in Q(o)$.

(34) Let A be a partial non-empty universal algebra structure and P be a partition of A. Suppose $P = \mathrm{Classes}\,\mathrm{LimDomRel}(A)$. Let S be a non void non empty many sorted signature. If A can be characterized by S, then $\mathrm{MSSign}(A,P)$ is rougher than S.

ACKNOWLEDGMENTS

I would like to thank Professor Andrzej Trybulec for suggesting me the problem solved here.

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Received October 1, 1995

Published January 2, 2004