JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Bounding Boxes for Compact Sets in $\mathcal{E}^2$

Czesław Byliński
Warsaw University
Białystok

Piotr Rudnicki
University of Alberta
Edmonton

**Summary.** We define pseudocompact topological spaces and prove that every compact space is pseudocompact. We also solve an exercise from [14] p.225 that for a topological space $X$ the following are equivalent:

- Every continuous real map from $X$ is bounded (i.e. $X$ is pseudocompact).
- Every continuous real map from $X$ attains minimum.
- Every continuous real map from $X$ attains maximum.

Finally, for a compact set in $E^2$ we define its bounding rectangle and introduce a collection of notions associated with the box.

MML Identifier: `PSCOMP_1`.

WWW: `http://mizar.org/JFM/Vol9/pscomp_1.html`

The articles [20], [23], [1], [22], [16], [7], [18], [10], [21], [24], [3], [4], [13], [12], [15], [11], [19], [17], [6], [5], [2], [8], and [9] provide the notation and terminology for this paper.

## 1. Preliminaries

Let $X$ be a set. Let us observe that $X$ has non empty elements if and only if:

(Def. 1) $0 \notin X$.

We introduce $X$ is without zero as a synonym of $X$ has non empty elements. We introduce $X$ has zero as an antonym of $X$ has non empty elements.

Let us mention that $\mathbb{R}$ has zero and $\mathbb{N}$ has zero.

Let us observe that there exists a set which is non empty and without zero and there exists a set which is non empty and has zero.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty and without zero and there exists a subset of $\mathbb{R}$ which is non empty and has zero.

We now state the proposition

(1) For every set $F$ such that $F$ is non empty and $\subseteq$-linear and has non empty elements holds $F$ is centered.

Let $F$ be a set. One can check that every family of subsets of $F$ which is non empty and $\subseteq$-linear and has non empty elements is also centered.

Let $A$, $B$ be sets and let $f$ be a function from $A$ into $B$. Then $\operatorname{rng} f$ is a subset of $B$.

Let $X$, $Y$ be non empty sets and let $f$ be a function from $X$ into $Y$. Note that $f^{\circ}X$ is non empty.

Let $X$, $Y$ be sets and let $f$ be a function from $X$ into $Y$. The functor ${}^{-1}f$ yields a function from $2^Y$ into $2^X$ and is defined by:

(Def. 2) For every subset $y$ of $Y$ holds $({}^{-1}f)(y) = f^{-1}(y)$.

We now state the proposition

(2) Let $X$, $Y$, $x$ be sets, $S$ be a subset of $2^Y$, and $f$ be a function from $X$ into $Y$. If $x \in \bigcap(({}^{-1}f)^\circ S)$, then $f(x) \in \bigcap S$.

In the sequel $r$, $s$, $t$ are real numbers.
One can prove the following propositions:

(3) If $|r| + |s| = 0$, then $r = 0$.

(4) If $r < s$ and $s < t$, then $|s| < |r| + |t|$.

(5) If $-s < r$ and $r < s$, then $|r| < s$.

In the sequel $s_1$ is a sequence of real numbers and $X$, $Y$ are subsets of $\mathbb{R}$.
We now state two propositions:

(6) If $s_1$ is convergent and non-zero and $\lim s_1 = 0$, then ${s_1}^{-1}$ is non bounded.

(7) $\operatorname{rng} s_1$ is bounded iff $s_1$ is bounded.

Let $X$ be a real-membered set. We introduce $\sup X$ as a synonym of $\sup X$. We introduce $\inf X$ as a synonym of $\inf X$.
Let $X$ be a subset of $\mathbb{R}$. Then $\sup X$ is an element of $\mathbb{R}$. Then $\inf X$ is an element of $\mathbb{R}$.
We now state several propositions:

(8) For every non empty real-membered set $X$ and for every $t$ such that for every $s$ such that $s \in X$ holds $s \geq t$ holds $\inf X \geq t$.

(9) Let $X$ be a non empty real-membered set. Suppose for every $s$ such that $s \in X$ holds $s \geq r$ and for every $t$ such that for every $s$ such that $s \in X$ holds $s \geq t$ holds $r \geq t$. Then $r = \inf X$.

(10) For every non empty real-membered set $X$ and for every $r$ and for every $t$ such that for every $s$ such that $s \in X$ holds $s \leq t$ holds $\sup X \leq t$.

(11) Let $X$ be a non empty real-membered set and given $r$. Suppose for every $s$ such that $s \in X$ holds $s \leq r$ and for every $t$ such that for every $s$ such that $s \in X$ holds $s \leq t$ holds $r \leq t$. Then $r = \sup X$.

(12) Let $X$ be a non empty real-membered set and $Y$ be a real-membered set. If $X \subseteq Y$ and $Y$ is lower bounded, then $\inf Y \leq \inf X$.

(13) Let $X$ be a non empty real-membered set and $Y$ be a real-membered set. If $X \subseteq Y$ and $Y$ is upper bounded, then $\sup X \leq \sup Y$.

Let $X$ be a real-membered set. We say that $X$ has maximum if and only if:

(Def. 3) $X$ is upper bounded and $\sup X \in X$.

We say that $X$ has minimum if and only if:

(Def. 4) $X$ is lower bounded and $\inf X \in X$.

One can check that there exists a subset of $\mathbb{R}$ which is non empty, closed, and bounded.
Let $R$ be a family of subsets of $\mathbb{R}$. We say that $R$ is open if and only if:

(Def. 5) For every subset $X$ of $\mathbb{R}$ such that $X \in R$ holds $X$ is open.

We say that $R$ is closed if and only if:

(Def. 6) For every subset $X$ of $\mathbb{R}$ such that $X \in R$ holds $X$ is closed.

In the sequel $r_3$, $r_1$, $r_2$, $q_3$, $p_3$ denote real numbers.

Let $X$ be a subset of $\mathbb{R}$. The functor $-X$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 7) $-X = \{-r_3 : r_3 \in X\}$.

Let us notice that the functor $-X$ is involutive.

We now state the proposition

(14) $r \in X$ iff $-r \in -X$.

Let $X$ be a non empty subset of $\mathbb{R}$. One can check that $-X$ is non empty.

Next we state several propositions:

(15) $X$ is upper bounded iff $-X$ is lower bounded.

(16) $X$ is lower bounded iff $-X$ is upper bounded.

(17) For every non empty subset $X$ of $\mathbb{R}$ such that $X$ is lower bounded holds $\inf X = -\sup(-X)$.

(18) For every non empty subset $X$ of $\mathbb{R}$ such that $X$ is upper bounded holds $\sup X = -\inf(-X)$.

(19) $X$ is closed iff $-X$ is closed.

Let $X$ be a subset of $\mathbb{R}$ and let $p$ be a real number. The functor $p+X$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 8) $p+X = \{p+r_3 : r_3 \in X\}$.

Next we state the proposition

(20) $r \in X$ iff $q_3+r \in q_3+X$.

Let $X$ be a non empty subset of $\mathbb{R}$ and let $s$ be a real number. One can verify that $s+X$ is non empty.

Next we state several propositions:

(21) $X = 0+X$.

(22) $q_3+(p_3+X) = (q_3+p_3)+X$.

(23) $X$ is upper bounded iff $q_3+X$ is upper bounded.

(24) $X$ is lower bounded iff $q_3+X$ is lower bounded.

(25) For every non empty subset $X$ of $\mathbb{R}$ such that $X$ is lower bounded holds $\inf(q_3+X) = q_3+\inf X$.

(26) For every non empty subset $X$ of $\mathbb{R}$ such that $X$ is upper bounded holds $\sup(q_3+X) = q_3+\sup X$.

(27) $X$ is closed iff $q_3+X$ is closed.

Let $X$ be a subset of $\mathbb{R}$. The functor $\mathrm{Inv}X$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 9) $\mathrm{Inv}X = \{\frac{1}{r_3} : r_3 \in X\}$.

The following proposition is true

(28) For every without zero subset $X$ of $\mathbb{R}$ holds $r \in X$ iff $\frac{1}{r} \in \mathrm{Inv}X$.

Let $X$ be a non empty without zero subset of $\mathbb{R}$. Note that $\mathrm{Inv}X$ is non empty and without zero.

Let $X$ be a without zero subset of $\mathbb{R}$. Observe that $\mathrm{Inv}X$ is without zero.

The following three propositions are true:

(29) For every without zero subset $X$ of $\mathbb{R}$ holds $\mathrm{Inv}\,\mathrm{Inv}\,X = X$.

(30) For every without zero subset $X$ of $\mathbb{R}$ such that $X$ is closed and bounded holds $\mathrm{Inv}\,X$ is closed.

(31) For every family $Z$ of subsets of $\mathbb{R}$ such that $Z$ is closed holds $\bigcap Z$ is closed.

Let $X$ be a subset of $\mathbb{R}$. The functor $\overline{X}$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 10) $\overline{X} = \bigcap\{A; A$ ranges over elements of $2^{\mathbb{R}}$: $X \subseteq A \wedge A$is closed$\}$.

Let us note that the functor $\overline{X}$ is projective.
Let $X$ be a subset of $\mathbb{R}$. One can check that $\overline{X}$ is closed.
One can prove the following propositions:

(32) For every closed subset $Y$ of $\mathbb{R}$ such that $X \subseteq Y$ holds $\overline{X} \subseteq Y$.

(33) $X \subseteq \overline{X}$.

(34) $X$ is closed iff $X = \overline{X}$.

(35) $\overline{\emptyset_{\mathbb{R}}} = \emptyset$.

(36) $\overline{\Omega_{\mathbb{R}}} = \mathbb{R}$.

(37) If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

(38) $r_3 \in \overline{X}$ iff for every open subset $O$ of $\mathbb{R}$ such that $r_3 \in O$ holds $O \cap X$ is non empty.

(39) If $r_3 \in \overline{X}$, then there exists $s_1$ such that $\mathrm{rng}\, s_1 \subseteq X$ and $s_1$ is convergent and $\lim s_1 = r_3$.

## 2. Functions into Reals

Let $X$ be a set and let $f$ be a function from $X$ into $\mathbb{R}$. Let us observe that $f$ is lower bounded if and only if:

(Def. 11) $f^{\circ}X$ is lower bounded.

Let us observe that $f$ is upper bounded if and only if:

(Def. 12) $f^{\circ}X$ is upper bounded.

Let $X$ be a set and let $f$ be a function from $X$ into $\mathbb{R}$. We say that $f$ has maximum if and only if:

(Def. 14)[1] $f^{\circ}X$ has maximum.

We say that $f$ has minimum if and only if:

(Def. 15) $f^{\circ}X$ has minimum.

Let $X$ be a set and let $f$ be a function from $X$ into $\mathbb{R}$. The functor $-f$ yielding a function from $X$ into $\mathbb{R}$ is defined as follows:

(Def. 16) For every set $p$ such that $p \in X$ holds $(-f)(p) = -f(p)$.

Let us note that the functor $-f$ is involutive.
One can prove the following propositions:

(40) For all sets $X, A$ and for every function $f$ from $X$ into $\mathbb{R}$ holds $(-f)^{\circ}A = -f^{\circ}A$.

(41) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{R}$ holds $f$ has minimum iff $-f$ has maximum.

[1] The definition (Def. 13) has been removed.

(42) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{R}$ holds $f$ has maximum iff $-f$ has minimum.

(43) For every set $X$ and for every subset $A$ of $\mathbb{R}$ and for every function $f$ from $X$ into $\mathbb{R}$ holds $(-f)^{-1}(A) = f^{-1}(-A)$.

Let $X$ be a set, let $r$ be a real number, and let $f$ be a function from $X$ into $\mathbb{R}$. The functor $r+f$ yields a function from $X$ into $\mathbb{R}$ and is defined as follows:

(Def. 17) For every set $p$ such that $p \in X$ holds $(r+f)(p) = r+f(p)$.

One can prove the following propositions:

(44) For all sets $X, A$ and for every function $f$ from $X$ into $\mathbb{R}$ and for every real number $s$ holds $(s+f)^\circ A = s+f^\circ A$.

(45) For every set $X$ and for every subset $A$ of $\mathbb{R}$ and for every function $f$ from $X$ into $\mathbb{R}$ and for every $q_3$ holds $(q_3+f)^{-1}(A) = f^{-1}(-q_3+A)$.

Let $X$ be a set and let $f$ be a function from $X$ into $\mathbb{R}$. The functor $\mathrm{Inv}\, f$ yields a function from $X$ into $\mathbb{R}$ and is defined as follows:

(Def. 18) For every set $p$ such that $p \in X$ holds $(\mathrm{Inv}\, f)(p) = \frac{1}{f(p)}$.

Let us observe that the functor $\mathrm{Inv}\, f$ is involutive.

The following proposition is true

(46) For every set $X$ and for every without zero subset $A$ of $\mathbb{R}$ and for every function $f$ from $X$ into $\mathbb{R}$ holds $(\mathrm{Inv}\, f)^{-1}(A) = f^{-1}(\mathrm{Inv} A)$.

## 3. REAL MAPS

Let $T$ be a 1-sorted structure. A real map of $T$ is a function from the carrier of $T$ into $\mathbb{R}$.

Let $T$ be a non empty 1-sorted structure. Observe that there exists a real map of $T$ which is bounded.

In this article we present several logical schemes. The scheme *NonUniqExRF* deals with a non empty topological structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a real map $f$ of $\mathcal{A}$ such that for every element $x$ of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following condition is met:

- For every set $x$ such that $x \in$ the carrier of $\mathcal{A}$ there exists $r_3$ such that $\mathcal{P}[x, r_3]$.

The scheme *LambdaRF* deals with a non empty topological structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

There exists a real map $f$ of $\mathcal{A}$ such that for every element $x$ of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

Let $T$ be a 1-sorted structure, let $f$ be a real map of $T$, and let $P$ be a set. Then $f^{-1}(P)$ is a subset of $T$.

Let $T$ be a 1-sorted structure and let $f$ be a real map of $T$. The functor $\inf f$ yielding a real number is defined by:

(Def. 20)[2] $\inf f = \inf(f^\circ(\text{the carrier of } T))$.

The functor $\sup f$ yields a real number and is defined by:

(Def. 21) $\sup f = \sup(f^\circ(\text{the carrier of } T))$.

The following propositions are true:

(47) Let $T$ be a non empty topological space, $f$ be a lower bounded real map of $T$, and $p$ be a point of $T$. Then $f(p) \geq \inf f$.

[2] The definition (Def. 19) has been removed.

(48) Let $T$ be a non empty topological space, $f$ be a lower bounded real map of $T$, and $s$ be a real number. If for every point $t$ of $T$ holds $f(t) \geq s$, then $\inf f \geq s$.

(49) Let $T$ be a non empty topological space and $f$ be a real map of $T$. Suppose for every point $p$ of $T$ holds $f(p) \geq r$ and for every $t$ such that for every point $p$ of $T$ holds $f(p) \geq t$ holds $r \geq t$. Then $r = \inf f$.

(50) Let $T$ be a non empty topological space, $f$ be an upper bounded real map of $T$, and $p$ be a point of $T$. Then $f(p) \leq \sup f$.

(51) Let $T$ be a non empty topological space, $f$ be an upper bounded real map of $T$, and given $t$. If for every point $p$ of $T$ holds $f(p) \leq t$, then $\sup f \leq t$.

(52) Let $T$ be a non empty topological space and $f$ be a real map of $T$. Suppose for every point $p$ of $T$ holds $f(p) \leq r$ and for every $t$ such that for every point $p$ of $T$ holds $f(p) \leq t$ holds $r \leq t$. Then $r = \sup f$.

(53) For every non empty 1-sorted structure $T$ and for every bounded real map $f$ of $T$ holds $\inf f \leq \sup f$.

Let $T$ be a topological structure and let $f$ be a real map of $T$. We say that $f$ is continuous if and only if:

(Def. 25)[3] For every subset $Y$ of $\mathbb{R}$ such that $Y$ is closed holds $f^{-1}(Y)$ is closed.

Let $T$ be a non empty topological space. Observe that there exists a real map of $T$ which is continuous.

Let $T$ be a non empty topological space and let $S$ be a non empty subspace of $T$. Observe that there exists a real map of $S$ which is continuous.

In the sequel $T$ denotes a topological structure and $f$ denotes a real map of $T$.

Next we state four propositions:

(54) $f$ is continuous iff for every subset $Y$ of $\mathbb{R}$ such that $Y$ is open holds $f^{-1}(Y)$ is open.

(55) If $f$ is continuous, then $-f$ is continuous.

(56) If $f$ is continuous, then $r_3 + f$ is continuous.

(57) If $f$ is continuous and $0 \notin \operatorname{rng} f$, then $\operatorname{Inv} f$ is continuous.

Let $X$, $Y$ be sets, let $f$ be a function from $2^X$ into $2^Y$, and let $R$ be a family of subsets of $X$. Then $f^\circ R$ is a family of subsets of $Y$.

The following two propositions are true:

(58) For every family $R$ of subsets of $\mathbb{R}$ such that $f$ is continuous and $R$ is open holds $({}^{-1}f)^\circ R$ is open.

(59) For every family $R$ of subsets of $\mathbb{R}$ such that $f$ is continuous and $R$ is closed holds $({}^{-1}f)^\circ R$ is closed.

Let $T$ be a non empty topological structure, let $X$ be a subset of $T$, and let $f$ be a real map of $T$. The functor $f \upharpoonright X$ yielding a real map of $T \restriction X$ is defined by:

(Def. 26) $f \upharpoonright X = f \restriction X$.

Let $T$ be a non empty topological space. Note that there exists a subset of $T$ which is compact and non empty.

Let $T$ be a non empty topological space, let $f$ be a continuous real map of $T$, and let $X$ be a subset of $T$. One can verify that $f \upharpoonright X$ is continuous.

Let $T$ be a non empty topological space and let $P$ be a compact non empty subset of $T$. One can check that $T \restriction P$ is compact.

[3] The definitions (Def. 22)–(Def. 24) have been removed.

## 4. PSEUDOCOMPACT SPACES

We now state two propositions:

(60) Let $T$ be a non empty topological space. Then for every real map $f$ of $T$ such that $f$ is continuous holds $f$ has maximum if and only if for every real map $f$ of $T$ such that $f$ is continuous holds $f$ has minimum.

(61) Let $T$ be a non empty topological space. Then for every real map $f$ of $T$ such that $f$ is continuous holds $f$ is bounded if and only if for every real map $f$ of $T$ such that $f$ is continuous holds $f$ has maximum.

Let $T$ be a topological structure. We say that $T$ is pseudocompact if and only if:

(Def. 27) For every real map $f$ of $T$ such that $f$ is continuous holds $f$ is bounded.

Let us observe that every non empty topological space which is compact is also pseudocompact.
Let us note that there exists a topological space which is compact and non empty.
Let $T$ be a pseudocompact non empty topological space. Note that every real map of $T$ which is continuous is also bounded and has maximum and minimum.
One can prove the following two propositions:

(62) Let $T$ be a non empty topological space, $X$ be a non empty subset of $T$, $Y$ be a compact subset of $T$, and $f$ be a continuous real map of $T$. If $X \subseteq Y$, then $\inf(f \restriction Y) \leq \inf(f \restriction X)$.

(63) Let $T$ be a non empty topological space, $X$ be a non empty subset of $T$, $Y$ be a compact subset of $T$, and $f$ be a continuous real map of $T$. If $X \subseteq Y$, then $\sup(f \restriction X) \leq \sup(f \restriction Y)$.

## 5. BOUNDING BOXES FOR COMPACT SETS IN $\mathcal{E}^2$

Let $n$ be a natural number and let $p_1$, $p_2$ be points of $\mathcal{E}^n_{\rm T}$. One can check that $\mathcal{L}(p_1, p_2)$ is compact.
We now state the proposition

(64) For every natural number $n$ and for all compact subsets $X$, $Y$ of $\mathcal{E}^n_{\rm T}$ holds $X \cap Y$ is compact.

In the sequel $p$ is a point of $\mathcal{E}^2_{\rm T}$, $P$ is a subset of $\mathcal{E}^2_{\rm T}$, $Z$ is a non empty subset of $\mathcal{E}^2_{\rm T}$, and $X$ is a non empty compact subset of $\mathcal{E}^2_{\rm T}$.
The real map proj1 of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 28) For every point $p$ of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{proj1}(p) = p_{\mathbf{1}}$.

The real map proj2 of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 29) For every point $p$ of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{proj2}(p) = p_{\mathbf{2}}$.

We now state four propositions:

(65) $\mathrm{proj1}^{-1}(]r,s[) = \{[r_1, r_2] : r < r_1 \wedge r_1 < s\}$.

(66) For all $r_3$, $q_3$ such that $P = \{[r_1, r_2] : r_3 < r_1 \wedge r_1 < q_3\}$ holds $P$ is open.

(67) $\mathrm{proj2}^{-1}(]r,s[) = \{[r_1, r_2] : r < r_2 \wedge r_2 < s\}$.

(68) For all $r_3$, $q_3$ such that $P = \{[r_1, r_2] : r_3 < r_2 \wedge r_2 < q_3\}$ holds $P$ is open.

One can check that proj1 is continuous and proj2 is continuous.
Next we state two propositions:

(69) For every subset $X$ of $\mathcal{E}^2_{\rm T}$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in X$ holds $(\mathrm{proj1} \restriction X)(p) = p_{\mathbf{1}}$.

(70) For every subset $X$ of $\mathcal{E}^2_{\mathrm{T}}$ and for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ holds $(\mathrm{proj2} \restriction X)(p) = p_{\mathbf{2}}$.

Let $X$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-bound$(X)$ yielding a real number is defined as follows:

(Def. 30) W-bound$(X) = \inf(\mathrm{proj1} \restriction X)$.

The functor N-bound$(X)$ yielding a real number is defined as follows:

(Def. 31) N-bound$(X) = \sup(\mathrm{proj2} \restriction X)$.

The functor E-bound$(X)$ yielding a real number is defined by:

(Def. 32) E-bound$(X) = \sup(\mathrm{proj1} \restriction X)$.

The functor S-bound$(X)$ yielding a real number is defined by:

(Def. 33) S-bound$(X) = \inf(\mathrm{proj2} \restriction X)$.

We now state the proposition

(71) If $p \in X$, then W-bound$(X) \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq$ E-bound$(X)$ and S-bound$(X) \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq$ N-bound$(X)$.

Let $X$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor SW-corner$(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 34) SW-corner$(X) = [$W-bound$(X),$S-bound$(X)]$.

The functor NW-corner$(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 35) NW-corner$(X) = [$W-bound$(X),$N-bound$(X)]$.

The functor NE-corner$(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 36) NE-corner$(X) = [$E-bound$(X),$N-bound$(X)]$.

The functor SE-corner$(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 37) SE-corner$(X) = [$E-bound$(X),$S-bound$(X)]$.

We now state a number of propositions:

(72) (SW-corner$(P))_{\mathbf{1}} =$ W-bound$(P)$.

(73) (SW-corner$(P))_{\mathbf{2}} =$ S-bound$(P)$.

(74) (NW-corner$(P))_{\mathbf{1}} =$ W-bound$(P)$.

(75) (NW-corner$(P))_{\mathbf{2}} =$ N-bound$(P)$.

(76) (NE-corner$(P))_{\mathbf{1}} =$ E-bound$(P)$.

(77) (NE-corner$(P))_{\mathbf{2}} =$ N-bound$(P)$.

(78) (SE-corner$(P))_{\mathbf{1}} =$ E-bound$(P)$.

(79) (SE-corner$(P))_{\mathbf{2}} =$ S-bound$(P)$.

(80) (SW-corner$(P))_{\mathbf{1}} =$ (NW-corner$(P))_{\mathbf{1}}$.

(81) (SE-corner$(P))_{\mathbf{1}} =$ (NE-corner$(P))_{\mathbf{1}}$.

(82) (NW-corner$(P))_{\mathbf{2}} =$ (NE-corner$(P))_{\mathbf{2}}$.

(83) (SW-corner$(P))_{\mathbf{2}} =$ (SE-corner$(P))_{\mathbf{2}}$.

Let $X$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{W}_{\mathrm{most}}(X)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 38) $\mathrm{W}_{\mathrm{most}}(X) = \mathcal{L}(\text{SW-corner}(X), \text{NW-corner}(X)) \cap X$.

The functor $\mathrm{N}_{\mathrm{most}}(X)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 39) $\mathrm{N}_{\mathrm{most}}(X) = \mathcal{L}(\text{NW-corner}(X), \text{NE-corner}(X)) \cap X$.

The functor $\mathrm{E}_{\mathrm{most}}(X)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 40) $\mathrm{E}_{\mathrm{most}}(X) = \mathcal{L}(\text{SE-corner}(X), \text{NE-corner}(X)) \cap X$.

The functor $\mathrm{S}_{\mathrm{most}}(X)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 41) $\mathrm{S}_{\mathrm{most}}(X) = \mathcal{L}(\text{SW-corner}(X), \text{SE-corner}(X)) \cap X$.

Let $X$ be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. One can verify the following observations:

* $\mathrm{W}_{\mathrm{most}}(X)$ is non empty and compact,
* $\mathrm{N}_{\mathrm{most}}(X)$ is non empty and compact,
* $\mathrm{E}_{\mathrm{most}}(X)$ is non empty and compact, and
* $\mathrm{S}_{\mathrm{most}}(X)$ is non empty and compact.

Let $X$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{W}_{\mathrm{min}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 42) $\mathrm{W}_{\mathrm{min}}(X) = [\text{W-bound}(X), \inf(\mathrm{proj2} \restriction \mathrm{W}_{\mathrm{most}}(X))]$.

The functor $\mathrm{W}_{\mathrm{max}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 43) $\mathrm{W}_{\mathrm{max}}(X) = [\text{W-bound}(X), \sup(\mathrm{proj2} \restriction \mathrm{W}_{\mathrm{most}}(X))]$.

The functor $\mathrm{N}_{\mathrm{min}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 44) $\mathrm{N}_{\mathrm{min}}(X) = [\inf(\mathrm{proj1} \restriction \mathrm{N}_{\mathrm{most}}(X)), \text{N-bound}(X)]$.

The functor $\mathrm{N}_{\mathrm{max}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 45) $\mathrm{N}_{\mathrm{max}}(X) = [\sup(\mathrm{proj1} \restriction \mathrm{N}_{\mathrm{most}}(X)), \text{N-bound}(X)]$.

The functor $\mathrm{E}_{\mathrm{max}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 46) $\mathrm{E}_{\mathrm{max}}(X) = [\text{E-bound}(X), \sup(\mathrm{proj2} \restriction \mathrm{E}_{\mathrm{most}}(X))]$.

The functor $\mathrm{E}_{\mathrm{min}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 47) $\mathrm{E}_{\mathrm{min}}(X) = [\text{E-bound}(X), \inf(\mathrm{proj2} \restriction \mathrm{E}_{\mathrm{most}}(X))]$.

The functor $\mathrm{S}_{\mathrm{max}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 48) $\mathrm{S}_{\mathrm{max}}(X) = [\sup(\mathrm{proj1} \restriction \mathrm{S}_{\mathrm{most}}(X)), \text{S-bound}(X)]$.

The functor $\mathrm{S}_{\mathrm{min}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 49) $\mathrm{S}_{\mathrm{min}}(X) = [\inf(\mathrm{proj1} \restriction \mathrm{S}_{\mathrm{most}}(X)), \text{S-bound}(X)]$.

One can prove the following propositions:

(84) $(\mathrm{W}_{\mathrm{min}}(P))_{\mathbf{1}} = \text{W-bound}(P)$ and $(\mathrm{W}_{\mathrm{max}}(P))_{\mathbf{1}} = \text{W-bound}(P)$.

(85) $(\text{SW-corner}(P))_{\mathbf{1}} = (\mathrm{W}_{\mathrm{min}}(P))_{\mathbf{1}}$ and $(\text{SW-corner}(P))_{\mathbf{1}} = (\mathrm{W}_{\mathrm{max}}(P))_{\mathbf{1}}$ and $(\mathrm{W}_{\mathrm{min}}(P))_{\mathbf{1}} = (\mathrm{W}_{\mathrm{max}}(P))_{\mathbf{1}}$ and $(\mathrm{W}_{\mathrm{min}}(P))_{\mathbf{1}} = (\text{NW-corner}(P))_{\mathbf{1}}$ and $(\mathrm{W}_{\mathrm{max}}(P))_{\mathbf{1}} = (\text{NW-corner}(P))_{\mathbf{1}}$.

(86) $(\mathrm{W}_{\mathrm{min}}(P))_{\mathbf{2}} = \inf(\mathrm{proj2} \restriction \mathrm{W}_{\mathrm{most}}(P))$ and $(\mathrm{W}_{\mathrm{max}}(P))_{\mathbf{2}} = \sup(\mathrm{proj2} \restriction \mathrm{W}_{\mathrm{most}}(P))$.

(87) $(\text{SW-corner}(X))_{\mathbf{2}} \leq (\mathrm{W}_{\mathrm{min}}(X))_{\mathbf{2}}$ and $(\text{SW-corner}(X))_{\mathbf{2}} \leq (\mathrm{W}_{\mathrm{max}}(X))_{\mathbf{2}}$ and $(\text{SW-corner}(X))_{\mathbf{2}} \leq (\text{NW-corner}(X))_{\mathbf{2}}$ and $(\mathrm{W}_{\mathrm{min}}(X))_{\mathbf{2}} \leq (\mathrm{W}_{\mathrm{max}}(X))_{\mathbf{2}}$ and $(\mathrm{W}_{\mathrm{min}}(X))_{\mathbf{2}} \leq (\text{NW-corner}(X))_{\mathbf{2}}$ and $(\mathrm{W}_{\mathrm{max}}(X))_{\mathbf{2}} \leq (\text{NW-corner}(X))_{\mathbf{2}}$.

(88) If $p \in \mathrm{W_{most}}(Z)$, then $p_{\mathbf{1}} = (\mathrm{W_{min}}(Z))_{\mathbf{1}}$ and if $Z$ is compact, then $(\mathrm{W_{min}}(Z))_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (\mathrm{W_{max}}(Z))_{\mathbf{2}}$.

(89) $\mathrm{W_{most}}(X) \subseteq \mathcal{L}(\mathrm{W_{min}}(X), \mathrm{W_{max}}(X))$.

(90) $\mathcal{L}(\mathrm{W_{min}}(X), \mathrm{W_{max}}(X)) \subseteq \mathcal{L}(\text{SW-corner}(X), \text{NW-corner}(X))$.

(91) $\mathrm{W_{min}}(X) \in \mathrm{W_{most}}(X)$ and $\mathrm{W_{max}}(X) \in \mathrm{W_{most}}(X)$.

(92) $\mathcal{L}(\text{SW-corner}(X), \mathrm{W_{min}}(X)) \cap X = \{\mathrm{W_{min}}(X)\}$ and $\mathcal{L}(\mathrm{W_{max}}(X), \text{NW-corner}(X)) \cap X = \{\mathrm{W_{max}}(X)\}$.

(93) If $\mathrm{W_{min}}(X) = \mathrm{W_{max}}(X)$, then $\mathrm{W_{most}}(X) = \{\mathrm{W_{min}}(X)\}$.

(94) $(\mathrm{N_{min}}(P))_{\mathbf{2}} = \text{N-bound}(P)$ and $(\mathrm{N_{max}}(P))_{\mathbf{2}} = \text{N-bound}(P)$.

(95) $(\text{NW-corner}(P))_{\mathbf{2}} = (\mathrm{N_{min}}(P))_{\mathbf{2}}$ and $(\text{NW-corner}(P))_{\mathbf{2}} = (\mathrm{N_{max}}(P))_{\mathbf{2}}$ and $(\mathrm{N_{min}}(P))_{\mathbf{2}} = (\mathrm{N_{max}}(P))_{\mathbf{2}}$ and $(\mathrm{N_{min}}(P))_{\mathbf{2}} = (\text{NE-corner}(P))_{\mathbf{2}}$ and $(\mathrm{N_{max}}(P))_{\mathbf{2}} = (\text{NE-corner}(P))_{\mathbf{2}}$.

(96) $(\mathrm{N_{min}}(P))_{\mathbf{1}} = \inf(\text{proj1} \restriction \mathrm{N_{most}}(P))$ and $(\mathrm{N_{max}}(P))_{\mathbf{1}} = \sup(\text{proj1} \restriction \mathrm{N_{most}}(P))$.

(97) $(\text{NW-corner}(X))_{\mathbf{1}} \leq (\mathrm{N_{min}}(X))_{\mathbf{1}}$ and $(\text{NW-corner}(X))_{\mathbf{1}} \leq (\mathrm{N_{max}}(X))_{\mathbf{1}}$ and $(\text{NW-corner}(X))_{\mathbf{1}} \leq (\text{NE-corner}(X))_{\mathbf{1}}$ and $(\mathrm{N_{min}}(X))_{\mathbf{1}} \leq (\mathrm{N_{max}}(X))_{\mathbf{1}}$ and $(\mathrm{N_{min}}(X))_{\mathbf{1}} \leq (\text{NE-corner}(X))_{\mathbf{1}}$ and $(\mathrm{N_{max}}(X))_{\mathbf{1}} \leq (\text{NE-corner}(X))_{\mathbf{1}}$.

(98) If $p \in \mathrm{N_{most}}(Z)$, then $p_{\mathbf{2}} = (\mathrm{N_{min}}(Z))_{\mathbf{2}}$ and if $Z$ is compact, then $(\mathrm{N_{min}}(Z))_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (\mathrm{N_{max}}(Z))_{\mathbf{1}}$.

(99) $\mathrm{N_{most}}(X) \subseteq \mathcal{L}(\mathrm{N_{min}}(X), \mathrm{N_{max}}(X))$.

(100) $\mathcal{L}(\mathrm{N_{min}}(X), \mathrm{N_{max}}(X)) \subseteq \mathcal{L}(\text{NW-corner}(X), \text{NE-corner}(X))$.

(101) $\mathrm{N_{min}}(X) \in \mathrm{N_{most}}(X)$ and $\mathrm{N_{max}}(X) \in \mathrm{N_{most}}(X)$.

(102) $\mathcal{L}(\text{NW-corner}(X), \mathrm{N_{min}}(X)) \cap X = \{\mathrm{N_{min}}(X)\}$ and $\mathcal{L}(\mathrm{N_{max}}(X), \text{NE-corner}(X)) \cap X = \{\mathrm{N_{max}}(X)\}$.

(103) If $\mathrm{N_{min}}(X) = \mathrm{N_{max}}(X)$, then $\mathrm{N_{most}}(X) = \{\mathrm{N_{min}}(X)\}$.

(104) $(\mathrm{E_{min}}(P))_{\mathbf{1}} = \text{E-bound}(P)$ and $(\mathrm{E_{max}}(P))_{\mathbf{1}} = \text{E-bound}(P)$.

(105) $(\text{SE-corner}(P))_{\mathbf{1}} = (\mathrm{E_{min}}(P))_{\mathbf{1}}$ and $(\text{SE-corner}(P))_{\mathbf{1}} = (\mathrm{E_{max}}(P))_{\mathbf{1}}$ and $(\mathrm{E_{min}}(P))_{\mathbf{1}} = (\mathrm{E_{max}}(P))_{\mathbf{1}}$ and $(\mathrm{E_{min}}(P))_{\mathbf{1}} = (\text{NE-corner}(P))_{\mathbf{1}}$ and $(\mathrm{E_{max}}(P))_{\mathbf{1}} = (\text{NE-corner}(P))_{\mathbf{1}}$.

(106) $(\mathrm{E_{min}}(P))_{\mathbf{2}} = \inf(\text{proj2} \restriction \mathrm{E_{most}}(P))$ and $(\mathrm{E_{max}}(P))_{\mathbf{2}} = \sup(\text{proj2} \restriction \mathrm{E_{most}}(P))$.

(107) $(\text{SE-corner}(X))_{\mathbf{2}} \leq (\mathrm{E_{min}}(X))_{\mathbf{2}}$ and $(\text{SE-corner}(X))_{\mathbf{2}} \leq (\mathrm{E_{max}}(X))_{\mathbf{2}}$ and $(\text{SE-corner}(X))_{\mathbf{2}} \leq (\text{NE-corner}(X))_{\mathbf{2}}$ and $(\mathrm{E_{min}}(X))_{\mathbf{2}} \leq (\mathrm{E_{max}}(X))_{\mathbf{2}}$ and $(\mathrm{E_{min}}(X))_{\mathbf{2}} \leq (\text{NE-corner}(X))_{\mathbf{2}}$ and $(\mathrm{E_{max}}(X))_{\mathbf{2}} \leq (\text{NE-corner}(X))_{\mathbf{2}}$.

(108) If $p \in \mathrm{E_{most}}(Z)$, then $p_{\mathbf{1}} = (\mathrm{E_{min}}(Z))_{\mathbf{1}}$ and if $Z$ is compact, then $(\mathrm{E_{min}}(Z))_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (\mathrm{E_{max}}(Z))_{\mathbf{2}}$.

(109) $\mathrm{E_{most}}(X) \subseteq \mathcal{L}(\mathrm{E_{min}}(X), \mathrm{E_{max}}(X))$.

(110) $\mathcal{L}(\mathrm{E_{min}}(X), \mathrm{E_{max}}(X)) \subseteq \mathcal{L}(\text{SE-corner}(X), \text{NE-corner}(X))$.

(111) $\mathrm{E_{min}}(X) \in \mathrm{E_{most}}(X)$ and $\mathrm{E_{max}}(X) \in \mathrm{E_{most}}(X)$.

(112) $\mathcal{L}(\text{SE-corner}(X), \mathrm{E_{min}}(X)) \cap X = \{\mathrm{E_{min}}(X)\}$ and $\mathcal{L}(\mathrm{E_{max}}(X), \text{NE-corner}(X)) \cap X = \{\mathrm{E_{max}}(X)\}$.

(113) If $\mathrm{E_{min}}(X) = \mathrm{E_{max}}(X)$, then $\mathrm{E_{most}}(X) = \{\mathrm{E_{min}}(X)\}$.

(114) $(\mathrm{S_{min}}(P))_{\mathbf{2}} = \text{S-bound}(P)$ and $(\mathrm{S_{max}}(P))_{\mathbf{2}} = \text{S-bound}(P)$.

(115) $(\text{SW-corner}(P))_{\mathbf{2}} = (\mathrm{S}_{\min}(P))_{\mathbf{2}}$ and $(\text{SW-corner}(P))_{\mathbf{2}} = (\mathrm{S}_{\max}(P))_{\mathbf{2}}$ and $(\mathrm{S}_{\min}(P))_{\mathbf{2}} = (\mathrm{S}_{\max}(P))_{\mathbf{2}}$ and $(\mathrm{S}_{\min}(P))_{\mathbf{2}} = (\text{SE-corner}(P))_{\mathbf{2}}$ and $(\mathrm{S}_{\max}(P))_{\mathbf{2}} = (\text{SE-corner}(P))_{\mathbf{2}}$.

(116) $(\mathrm{S}_{\min}(P))_{\mathbf{1}} = \inf(\mathrm{proj1} \restriction \mathrm{S}_{\mathrm{most}}(P))$ and $(\mathrm{S}_{\max}(P))_{\mathbf{1}} = \sup(\mathrm{proj1} \restriction \mathrm{S}_{\mathrm{most}}(P))$.

(117) $(\text{SW-corner}(X))_{\mathbf{1}} \leq (\mathrm{S}_{\min}(X))_{\mathbf{1}}$ and $(\text{SW-corner}(X))_{\mathbf{1}} \leq (\mathrm{S}_{\max}(X))_{\mathbf{1}}$ and $(\text{SW-corner}(X))_{\mathbf{1}} \leq (\text{SE-corner}(X))_{\mathbf{1}}$ and $(\mathrm{S}_{\min}(X))_{\mathbf{1}} \leq (\mathrm{S}_{\max}(X))_{\mathbf{1}}$ and $(\mathrm{S}_{\min}(X))_{\mathbf{1}} \leq (\text{SE-corner}(X))_{\mathbf{1}}$ and $(\mathrm{S}_{\max}(X))_{\mathbf{1}} \leq (\text{SE-corner}(X))_{\mathbf{1}}$.

(118) If $p \in \mathrm{S}_{\mathrm{most}}(Z)$, then $p_{\mathbf{2}} = (\mathrm{S}_{\min}(Z))_{\mathbf{2}}$ and if $Z$ is compact, then $(\mathrm{S}_{\min}(Z))_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (\mathrm{S}_{\max}(Z))_{\mathbf{1}}$.

(119) $\mathrm{S}_{\mathrm{most}}(X) \subseteq \mathcal{L}(\mathrm{S}_{\min}(X), \mathrm{S}_{\max}(X))$.

(120) $\mathcal{L}(\mathrm{S}_{\min}(X), \mathrm{S}_{\max}(X)) \subseteq \mathcal{L}(\text{SW-corner}(X), \text{SE-corner}(X))$.

(121) $\mathrm{S}_{\min}(X) \in \mathrm{S}_{\mathrm{most}}(X)$ and $\mathrm{S}_{\max}(X) \in \mathrm{S}_{\mathrm{most}}(X)$.

(122) $\mathcal{L}(\text{SW-corner}(X), \mathrm{S}_{\min}(X)) \cap X = \{\mathrm{S}_{\min}(X)\}$ and $\mathcal{L}(\mathrm{S}_{\max}(X), \text{SE-corner}(X)) \cap X = \{\mathrm{S}_{\max}(X)\}$.

(123) If $\mathrm{S}_{\min}(X) = \mathrm{S}_{\max}(X)$, then $\mathrm{S}_{\mathrm{most}}(X) = \{\mathrm{S}_{\min}(X)\}$.

(124) If $\mathrm{W}_{\max}(P) = \mathrm{N}_{\min}(P)$, then $\mathrm{W}_{\max}(P) = \text{NW-corner}(P)$.

(125) If $\mathrm{N}_{\max}(P) = \mathrm{E}_{\max}(P)$, then $\mathrm{N}_{\max}(P) = \text{NE-corner}(P)$.

(126) If $\mathrm{E}_{\min}(P) = \mathrm{S}_{\max}(P)$, then $\mathrm{E}_{\min}(P) = \text{SE-corner}(P)$.

(127) If $\mathrm{S}_{\min}(P) = \mathrm{W}_{\min}(P)$, then $\mathrm{S}_{\min}(P) = \text{SW-corner}(P)$.

## REFERENCES


[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/pcomps_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/compts_1.html`.

[6] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_2.html`.

[7] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[8] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/euclid.html`.

[9] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/topreal1.html`.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[11] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_4.html`.

[12] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_2.html`.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_1.html`.

[14] M.G. Murdeshwar. *General Topology*. Wiley Eastern, 1990.

[15] Yatsuka Nakamura and Andrzej Trybulec. A mathematical model of CPU. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/ami_1.html`.

[16] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[17] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[18] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[21] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol16/membered.html.

[22] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[23] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[24] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

*Received July 29, 1997*

*Published January 2, 2004*