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On Projections in Projective Planes — Part II[1]

Eugeniusz Kusak
Warsaw University
Białystok

Wojciech Leończuk
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. We study in greater details projectivities on Desarguesian projective planes. We are particularly interested in the situation when the composition of given two projectivities can be replaced by another two, with given axis or centre of one of them.

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The articles [4], [6], [7], [8], [5], [3], [1], and [2] provide the notation and terminology for this paper.

In this paper I_1 is a projective space defined in terms of incidence and z is a point of I_1.

Let us consider I_1 and let A, B, C be lines of I_1. We say that A, B, C are concurrent if and only if:

(Def. 1) There exists an element o of the points of I_1 such that o lies on A and o lies on B and o lies on C.

Let us consider I_1 and let Z be a line of I_1. The functor $\mathrm{chain}(Z)$ yielding a subset of the points of I_1 is defined as follows:

(Def. 2) $\mathrm{chain}(Z) = \{z : z \text{ lies on } Z\}$.

For simplicity, we follow the rules: I_2 is a Desarguesian 2-dimensional projective space defined in terms of incidence, a, b, c, d, p, p'_1, q, o, o', o'', o'_1 are points of I_2, r, s, x, y, o_1, o_2 are points of I_2, and O_1, O_2, O_3, A, B, C, O, Q, R, S are lines of I_2.

Let us consider I_2. A partial function from the points of I_2 to the points of I_2 is said to be a projection of I_2 if:

(Def. 3) There exist a, A, B such that a does not lie on A and a does not lie on B and it $= \pi_a(A \to B)$.

The following propositions are true:

(1) If $A = B$ or $B = C$ or $C = A$, then A, B, C are concurrent.

(2) Suppose A, B, C are concurrent. Then

(i) A, C, B are concurrent,

(ii) B, A, C are concurrent,

(iii) B, C, A are concurrent,

(iv) C, A, B are concurrent, and

(v) C, B, A are concurrent.

[1] Supported by RPBP.III-24.C6.

(3) If o does not lie on A and o does not lie on B and y lies on B, then there exists x such that x lies on A and $\pi_o(A \to B)(x) = y$.

(5)[1] If o does not lie on A and o does not lie on B, then $\operatorname{dom} \pi_o(A \to B) = \operatorname{chain}(A)$.

(6) If o does not lie on A and o does not lie on B, then $\operatorname{rng} \pi_o(A \to B) = \operatorname{chain}(B)$.

(7) For every set x holds $x \in \operatorname{chain}(A)$ iff there exists a such that $x = a$ and a lies on A.

(8) If o does not lie on A and o does not lie on B, then $\pi_o(A \to B)$ is one-to-one.

(9) If o does not lie on A and o does not lie on B, then $\pi_o(A \to B)^{-1} = \pi_o(B \to A)$.

(10) For every projection f of I_2 holds f^{-1} is a projection of I_2.

(11) If o does not lie on A, then $\pi_o(A \to A) = \mathrm{id}_{\operatorname{chain}(A)}$.

(12) $\mathrm{id}_{\operatorname{chain}(A)}$ is a projection of I_2.

(13) If o does not lie on A and o does not lie on B and o does not lie on C, then $\pi_o(C \to B) \cdot \pi_o(A \to C) = \pi_o(A \to B)$.

(14) Suppose o_1 does not lie on O_1 and o_1 does not lie on O_2 and o_2 does not lie on O_2 and o_2 does not lie on O_3 and O_1, O_2, O_3 are concurrent and $O_1 \neq O_3$. Then there exists o such that o does not lie on O_1 and o does not lie on O_3 and $\pi_{o_2}(O_2 \to O_3) \cdot \pi_{o_1}(O_1 \to O_2) = \pi_o(O_1 \to O_3)$.

(15) Suppose that a does not lie on A and b does not lie on B and a does not lie on C and b does not lie on C and A, B, C are not concurrent and c lies on A and c lies on C and c lies on Q and b does not lie on Q and $A \neq Q$ and $a \neq b$ and $b \neq q$ and a lies on O and b lies on O and B, C, O are not concurrent and d lies on C and d lies on B and a lies on O_1 and d lies on O_1 and p lies on A and p lies on O_1 and q lies on O and q lies on O_2 and p lies on O_2 and p'_1 lies on O_2 and d lies on O_3 and b lies on O_3 and p'_1 lies on O_3 and p'_1 lies on Q and $Q \neq C$ and $q \neq a$ and q does not lie on A and q does not lie on Q. Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(16) Suppose that a does not lie on A and a does not lie on C and b does not lie on B and b does not lie on C and b does not lie on Q and A, B, C are not concurrent and $a \neq b$ and $b \neq q$ and $A \neq Q$ and c, o lie on A and o, o'', d lie on B and c, d, o' lie on C and a, b, d lie on O and c, o'_1 lie on Q and a, o, o' lie on O_1 and b, o', o'_1 lie on O_2 and o, o'_1, q lie on O_3 and q lies on O. Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(17) Suppose that a does not lie on A and a does not lie on C and b does not lie on B and b does not lie on C and b does not lie on Q and A, B, C are not concurrent and B, C, O are not concurrent and $A \neq Q$ and $Q \neq C$ and $a \neq b$ and c, p lie on A and d lies on B and c, d lie on C and a, b, q lie on O and c, p'_1 lie on Q and a, d, p lie on O_1 and q, p, p'_1 lie on O_2 and b, d, p'_1 lie on O_3. Then $q \neq a$ and $q \neq b$ and q does not lie on A and q does not lie on Q.

(18) Suppose that a does not lie on A and a does not lie on C and b does not lie on B and b does not lie on C and b does not lie on Q and A, B, C are not concurrent and $a \neq b$ and $A \neq Q$ and c, o lie on A and o, o'', d lie on B and c, d, o' lie on C and a, b, d lie on O and c, o'_1 lie on Q and a, o, o' lie on O_1 and b, o', o'_1 lie on O_2 and o, o'_1, q lie on O_3 and q lies on O. Then q does not lie on A and q does not lie on Q and $b \neq q$.

(19) Suppose that a does not lie on A and a does not lie on C and b does not lie on B and b does not lie on C and q does not lie on A and A, B, C are not concurrent and B, C, O are not concurrent and $a \neq b$ and $b \neq q$ and $q \neq a$ and c, p lie on A and d lies on B and c, d lie on C and a, b, q lie on O and c, p'_1 lie on Q and a, d, p lie on O_1 and q, p, p'_1 lie on O_2 and b, d, p'_1 lie on O_3. Then $Q \neq A$ and $Q \neq C$ and q does not lie on Q and b does not lie on Q.

[1] The proposition (4) has been removed.

(20) Suppose that a does not lie on A and a does not lie on C and b does not lie on B and b does not lie on C and q does not lie on A and A, B, C are not concurrent and $a \neq b$ and $b \neq q$ and c, o lie on A and o, o'', d lie on B and c, d, o' lie on C and a, b, d lie on O and c, o'_1 lie on Q and a, o, o' lie on O_1 and b, o', o'_1 lie on O_2 and o, o'_1, q lie on O_3 and q lies on O. Then b does not lie on Q and q does not lie on Q and $A \neq Q$.

(21) Suppose that a does not lie on A and b does not lie on B and a does not lie on C and b does not lie on C and A, B, C are not concurrent and A, C, Q are concurrent and b does not lie on Q and $A \neq Q$ and $a \neq b$ and a lies on O and b lies on O. Then there exists q such that q lies on O and q does not lie on A and q does not lie on Q and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(22) Suppose that a does not lie on A and b does not lie on B and a does not lie on C and b does not lie on C and A, B, C are not concurrent and B, C, Q are concurrent and a does not lie on Q and $B \neq Q$ and $a \neq b$ and a lies on O and b lies on O. Then there exists q such that q lies on O and q does not lie on B and q does not lie on Q and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_q(Q \to B) \cdot \pi_a(A \to Q)$.

(23) Suppose that a does not lie on A and b does not lie on B and a does not lie on C and b does not lie on C and a does not lie on B and b does not lie on A and c lies on A and c lies on C and d lies on B and d lies on C and a lies on S and d lies on S and c lies on R and b lies on R and s lies on A and s lies on S and r lies on B and r lies on R and s lies on Q and r lies on Q and A, B, C are not concurrent. Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_a(Q \to B) \cdot \pi_b(A \to Q)$.

(24) Suppose that a does not lie on A and b does not lie on B and a does not lie on C and b does not lie on C and $a \neq b$ and a lies on O and b lies on O and q lies on O and q does not lie on A and $q \neq b$ and A, B, C are not concurrent. Then there exists Q such that A, C, Q are concurrent and b does not lie on Q and q does not lie on Q and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(25) Suppose that a does not lie on A and b does not lie on B and a does not lie on C and b does not lie on C and $a \neq b$ and a lies on O and b lies on O and q lies on O and q does not lie on B and $q \neq a$ and A, B, C are not concurrent. Then there exists Q such that B, C, Q are concurrent and a does not lie on Q and q does not lie on Q and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_q(Q \to B) \cdot \pi_a(A \to Q)$.

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