JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Probability

Andrzej Nędzusiak
Warsaw University
Białystok

Summary. Some further theorems concerning probability, among them the equivalent definition of probability are discussed, followed by notions of independence of events and conditional probability and basic theorems on them.

MML Identifier: PROB_2.

WWW: http://mizar.org/JFM/Vol2/prob_2.html

The articles [10], [5], [12], [2], [11], [13], [6], [3], [4], [8], [7], [9], and [1] provide the notation and terminology for this paper.

For simplicity, we use the following convention: O_1 is a non empty set, m, n are natural numbers, x, y are sets, r, r_1, r_2, r_3 are real numbers, s_1, s_2 are sequences of real numbers, S_1 is a σ-field of subsets of O_1, A_1, B_1 are sequences of subsets of S_1, P, P_1 are probabilities on S_1, and A, B, C, A_2, A_3, A_4 are events of S_1.

One can prove the following three propositions:

(4)[1] For all r, r_1, r_2, r_3 such that $r \neq 0$ and $r_1 \neq 0$ holds $\frac{r_3}{r_1} = \frac{r_2}{r}$ iff $r_3 \cdot r = r_2 \cdot r_1$.

(5) If s_1 is convergent and for every n holds $s_2(n) = r - s_1(n)$, then s_2 is convergent and $\lim s_2 = r - \lim s_1$.

(6) $A \cap O_1 = A$ and $A \cap \Omega_{(S_1)} = A$.

The scheme *SeqExProb* deals with a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n) = \mathcal{F}(n)$

for all values of the parameter.

Let us consider O_1, S_1, A_1, n. Then $A_1(n)$ is an event of S_1.

Let us consider O_1, S_1, A_1. The functor $\bigcap A_1$ yields an event of S_1 and is defined as follows:

(Def. 1) $\bigcap A_1 = \operatorname{Intersection} A_1$.

We now state several propositions:

(9)[2] There exists B_1 such that for every n holds $B_1(n) = A_1(n) \cap B$.

(10) If A_1 is non-increasing and for every n holds $B_1(n) = A_1(n) \cap B$, then B_1 is non-increasing.

(11) For every function f from S_1 into $\mathbb{R}$ holds $(f \cdot A_1)(n) = f(A_1(n))$.

(12) If for every n holds $B_1(n) = A_1(n) \cap B$, then $\operatorname{Intersection} A_1 \cap B = \operatorname{Intersection} B_1$.

[1] The propositions (1)–(3) have been removed.
[2] The propositions (7) and (8) have been removed.

(13) If for every A holds $P(A) = P_1(A)$, then $P = P_1$.

(14) For every sequence A_1 of subsets of O_1 holds A_1 is non-increasing iff for every n holds $A_1(n+1) \subseteq A_1(n)$.

(15) For every sequence A_1 of subsets of O_1 holds A_1 is non-decreasing iff for every n holds $A_1(n) \subseteq A_1(n+1)$.

(16) For all sequences A_1, B_1 of subsets of O_1 such that for every n holds $A_1(n) = B_1(n)$ holds $A_1 = B_1$.

(17) For every sequence A_1 of subsets of O_1 holds A_1 is non-increasing iff ComplementA_1 is non-decreasing.

Let us consider O_1, S_1, A_1. The functor $A_1{}^{\mathbf{c}}$ yielding a sequence of subsets of S_1 is defined by:

(Def. 2) $A_1{}^{\mathbf{c}} = \mathrm{Complement} A_1$.

Let F be a function. We say that F is disjoint valued if and only if:

(Def. 3) If $x \neq y$, then $F(x)$ misses $F(y)$.

Let us consider O_1, S_1, A_1. Let us observe that A_1 is disjoint valued if and only if:

(Def. 4) For all m, n such that $m \neq n$ holds $A_1(m)$ misses $A_1(n)$.

One can prove the following propositions:

(20)[3] Let P be a function from S_1 into $\mathbb{R}$. Then P is a probability on S_1 if and only if the following conditions are satisfied:

(i) for every A holds $0 \leq P(A)$,

(ii) $P(O_1) = 1$,

(iii) for all A, B such that A misses B holds $P(A \cup B) = P(A) + P(B)$, and

(iv) for every A_1 such that A_1 is non-decreasing holds $P \cdot A_1$ is convergent and $\lim(P \cdot A_1) = P(\bigcup A_1)$.

(21) $P(A \cup B \cup C) = ((P(A) + P(B) + P(C)) - (P(A \cap B) + P(B \cap C) + P(A \cap C))) + P(A \cap B \cap C)$.

(22) $P(A \setminus A \cap B) = P(A) - P(A \cap B)$.

(23) $P(A \cap B) \leq P(B)$ and $P(A \cap B) \leq P(A)$.

(24) If $C = B^{\mathrm{c}}$, then $P(A) = P(A \cap B) + P(A \cap C)$.

(25) $(P(A) + P(B)) - 1 \leq P(A \cap B)$.

(26) $P(A) = 1 - P(\Omega_{(S_1)} \setminus A)$.

(27) $P(A) < 1$ iff $0 < P(\Omega_{(S_1)} \setminus A)$.

(28) $P(\Omega_{(S_1)} \setminus A) < 1$ iff $0 < P(A)$.

Let us consider O_1, S_1, P, A, B. We say that A and B are independent w.r.t P if and only if:

(Def. 5) $P(A \cap B) = P(A) \cdot P(B)$.

Let us consider C. We say that A, B and C are independent w.r.t P if and only if:

(Def. 6) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ and $P(A \cap B) = P(A) \cdot P(B)$ and $P(A \cap C) = P(A) \cdot P(C)$ and $P(B \cap C) = P(B) \cdot P(C)$.

[3] The propositions (18) and (19) have been removed.

We now state a number of propositions:

(31)[4] A and B are independent w.r.t P iff B and A are independent w.r.t P.

(32) A, B and C are independent w.r.t P if and only if the following conditions are satisfied:

(i) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$,

(ii) A and B are independent w.r.t P,

(iii) B and C are independent w.r.t P, and

(iv) A and C are independent w.r.t P.

(33) If A, B and C are independent w.r.t P, then B, A and C are independent w.r.t P.

(34) If A, B and C are independent w.r.t P, then A, C and B are independent w.r.t P.

(35) For every event E of S_1 such that $E = \emptyset$ holds A and E are independent w.r.t P.

(36) A and $\Omega_{(S_1)}$ are independent w.r.t P.

(37) For all A, B, P such that A and B are independent w.r.t P holds A and $\Omega_{(S_1)} \setminus B$ are independent w.r.t P.

(38) If A and B are independent w.r.t P, then $\Omega_{(S_1)} \setminus A$ and $\Omega_{(S_1)} \setminus B$ are independent w.r.t P.

(39) Let given A, B, C, P. Suppose A and B are independent w.r.t P and A and C are independent w.r.t P and B misses C. Then A and $B \cup C$ are independent w.r.t P.

(40) For all P, A, B such that A and B are independent w.r.t P and $P(A) < 1$ and $P(B) < 1$ holds $P(A \cup B) < 1$.

Let us consider O_1, S_1, P, B. Let us assume that $0 < P(B)$. The functor $(P|B)$ yields a probability on S_1 and is defined as follows:

(Def. 7) For every A holds $(P|B)(A) = \frac{P(A \cap B)}{P(B)}$.

The following propositions are true:

(42)[5] For all P, B, A such that $0 < P(B)$ holds $P(A \cap B) = (P|B)(A) \cdot P(B)$.

(43) For all P, A, B, C such that $0 < P(A \cap B)$ holds $P(A \cap B \cap C) = P(A) \cdot (P|A)(B) \cdot (P|(A \cap B))(C)$.

(44) For all P, A, B, C such that $C = B^{\rm c}$ and $0 < P(B)$ and $0 < P(C)$ holds $P(A) = (P|B)(A) \cdot P(B) + (P|C)(A) \cdot P(C)$.

(45) Let given P, A, A_2, A_3, A_4. Suppose A_2 misses A_3 and $A_4 = (A_2 \cup A_3)^{\rm c}$ and $0 < P(A_2)$ and $0 < P(A_3)$ and $0 < P(A_4)$. Then $P(A) = (P|A_2)(A) \cdot P(A_2) + (P|A_3)(A) \cdot P(A_3) + (P|A_4)(A) \cdot P(A_4)$.

(46) For all P, A, B such that $0 < P(B)$ holds $(P|B)(A) = P(A)$ iff A and B are independent w.r.t P.

(47) For all P, A, B such that $0 < P(B)$ and $P(B) < 1$ and $(P|B)(A) = (P|(\Omega_{(S_1)} \setminus B))(A)$ holds A and B are independent w.r.t P.

(48) For all P, A, B such that $0 < P(B)$ holds $\frac{(P(A)+P(B))-1}{P(B)} \leq (P|B)(A)$.

(49) For all A, B, P such that $0 < P(A)$ and $0 < P(B)$ holds $(P|B)(A) = \frac{(P|A)(B) \cdot P(A)}{P(B)}$.

[4] The propositions (29) and (30) have been removed.
[5] The proposition (41) has been removed.

(50) Let given B, A_2, A_3, P. Suppose $0 < P(B)$ and $A_3 = {A_2}^{\rm c}$ and $0 < P(A_2)$ and $0 < P(A_3)$. Then $(P|B)(A_2) = \frac{(P|A_2)(B) \cdot P(A_2)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3)}$ and $(P|B)(A_3) = \frac{(P|A_3)(B) \cdot P(A_3)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3)}$.

(51) Let given B, A_2, A_3, A_4, P. Suppose $0 < P(B)$ and $0 < P(A_2)$ and $0 < P(A_3)$ and $0 < P(A_4)$ and A_2 misses A_3 and $A_4 = (A_2 \cup A_3)^{\rm c}$. Then

(i) $(P|B)(A_2) = \frac{(P|A_2)(B) \cdot P(A_2)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3) + (P|A_4)(B) \cdot P(A_4)}$,

(ii) $(P|B)(A_3) = \frac{(P|A_3)(B) \cdot P(A_3)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3) + (P|A_4)(B) \cdot P(A_4)}$, and

(iii) $(P|B)(A_4) = \frac{(P|A_4)(B) \cdot P(A_4)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3) + (P|A_4)(B) \cdot P(A_4)}$.

(52) For all A, B, P such that $0 < P(B)$ holds $1 - \frac{P(\Omega_{(S_1)} \setminus A)}{P(B)} \leq (P|B)(A)$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_2.html`.

[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_1.html`.

[9] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/prob_1.html`.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received June 1, 1990

Published January 2, 2004