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Integer and Rational Exponents

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Summary. The article includes definitios and theorems which are needed to define real exponent. The following notions are defined: natural exponent, integer exponent and rational exponent.

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The articles [14], [2], [10], [4], [9], [1], [8], [3], [7], [6], [13], [12], [11], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: a, b, c are real numbers, m, n are natural numbers, k, l, i are integers, p, q are rational numbers, and s_1, s_2 are sequences of real numbers.

Next we state two propositions:

(2)[1] If s_1 is convergent and for every n holds $s_1(n) \geq a$, then $\lim s_1 \geq a$.

(3) If s_1 is convergent and for every n holds $s_1(n) \leq a$, then $\lim s_1 \leq a$.

Let a be a real number. The functor $(a^{\kappa})_{\kappa \in \mathbb{N}}$ yields a sequence of real numbers and is defined as follows:

(Def. 1) For every m holds $(a^{\kappa})_{\kappa \in \mathbb{N}}(m) = a^m$.

We now state two propositions:

(4) $s_1 = (a^{\kappa})_{\kappa \in \mathbb{N}}$ iff $s_1(0) = 1$ and for every m holds $s_1(m+1) = s_1(m) \cdot a$.

(5) For every a such that $a \neq 0$ and for every m holds $(a^{\kappa})_{\kappa \in \mathbb{N}}(m) \neq 0$.

Let a be a real number and let us consider n. Then a^n is a real number.

One can prove the following propositions:

(12)[2] If $0 \neq a$, then $0 \neq a^n$.

(13) If $0 < a$, then $0 < a^n$.

(14) $(\frac{1}{a})^n = \frac{1}{a^n}$.

(15) $(\frac{b}{a})^n = \frac{b^n}{a^n}$.

(17)[3] If $0 < a$ and $a \leq b$, then $a^n \leq b^n$.

[1] The proposition (1) has been removed.
[2] The propositions (6)–(11) have been removed.
[3] The proposition (16) has been removed.

(18) If $0 \leq a$ and $a < b$ and $1 \leq n$, then $a^n < b^n$.

(19) If $a \geq 1$, then $a^n \geq 1$.

(20) If $1 \leq a$ and $1 \leq n$, then $a \leq a^n$.

(21) If $1 < a$ and $2 \leq n$, then $a < a^n$.

(22) If $0 < a$ and $a \leq 1$ and $1 \leq n$, then $a^n \leq a$.

(23) If $0 < a$ and $a < 1$ and $2 \leq n$, then $a^n < a$.

(24) If $-1 < a$, then $(1+a)^n \geq 1 + n \cdot a$.

(25) If $0 < a$ and $a < 1$, then $(1+a)^n \leq 1 + 3^n \cdot a$.

(26) If s_1 is convergent and for every n holds $s_2(n) = s_1(n)^m$, then s_2 is convergent and $\lim s_2 = (\lim s_1)^m$.

Let us consider n and let a be a real number. Let us assume that $1 \leq n$. The functor $\mathrm{root}_n(a)$ yielding a real number is defined as follows:

(Def. 3)[4] (i) $(\mathrm{root}_n(a))^n = a$ and $\mathrm{root}_n(a) > 0$ if $a > 0$,

(ii) $\mathrm{root}_n(a) = 0$ if $a = 0$.

Let us consider n and let a be a real number. Then $\mathrm{root}_n(a)$ is a real number.

The following propositions are true:

(28)[5] If $a \geq 0$ and $n \geq 1$, then $(\mathrm{root}_n(a))^n = a$ and $\mathrm{root}_n(a^n) = a$.

(29) If $n \geq 1$, then $\mathrm{root}_n(1) = 1$.

(30) If $a \geq 0$, then $\mathrm{root}_1(a) = a$.

(31) If $a \geq 0$ and $b \geq 0$ and $n \geq 1$, then $\mathrm{root}_n(a \cdot b) = (\mathrm{root}_n(a)) \cdot (\mathrm{root}_n(b))$.

(32) If $a > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{1}{a}) = \frac{1}{\mathrm{root}_n(a)}$.

(33) If $a \geq 0$ and $b > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{a}{b}) = \frac{\mathrm{root}_n(a)}{\mathrm{root}_n(b)}$.

(34) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $\mathrm{root}_n(\mathrm{root}_m(a)) = \mathrm{root}_{n \cdot m}(a)$.

(35) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $(\mathrm{root}_n(a)) \cdot (\mathrm{root}_m(a)) = \mathrm{root}_{n \cdot m}(a^{n+m})$.

(36) If $0 \leq a$ and $a \leq b$ and $n \geq 1$, then $\mathrm{root}_n(a) \leq \mathrm{root}_n(b)$.

(37) If $a \geq 0$ and $a < b$ and $n \geq 1$, then $\mathrm{root}_n(a) < \mathrm{root}_n(b)$.

(38) If $a \geq 1$ and $n \geq 1$, then $\mathrm{root}_n(a) \geq 1$ and $a \geq \mathrm{root}_n(a)$.

(39) If $0 \leq a$ and $a < 1$ and $n \geq 1$, then $a \leq \mathrm{root}_n(a)$ and $\mathrm{root}_n(a) < 1$.

(40) If $a > 0$ and $n \geq 1$, then $(\mathrm{root}_n(a)) - 1 \leq \frac{a-1}{n}$.

(41) If $a \geq 0$, then $\mathrm{root}_2(a) = \sqrt{a}$.

(42) Let s be a sequence of real numbers and given a. Suppose $a > 0$ and for every n such that $n \geq 1$ holds $s(n) = \mathrm{root}_n(a)$. Then s is convergent and $\lim s = 1$.

Let a be a real number and let us consider k. The functor $a_{\mathbb{Z}}^k$ is defined by:

[4] The definition (Def. 2) has been removed.

[5] The proposition (27) has been removed.

(Def. 4)(i) $a^{k}_{\mathbb{Z}} = a^{|k|}$ if $k \geq 0$,

(ii) $a^{k}_{\mathbb{Z}} = (a^{|k|})^{-1}$ if $k < 0$.

Let a be a real number and let us consider k. Note that $a^{k}_{\mathbb{Z}}$ is real.
Let a be a real number and let us consider k. Then $a^{k}_{\mathbb{Z}}$ is a real number.
We now state a number of propositions:

(44)[6] $a^{0}_{\mathbb{Z}} = 1$.

(45) $a^{1}_{\mathbb{Z}} = a$.

(46) If $a \neq 0$ and $i = n$, then $a^{i}_{\mathbb{Z}} = a^{n}$.

(47) $1^{k}_{\mathbb{Z}} = 1$.

(48) If $a \neq 0$, then $a^{k}_{\mathbb{Z}} \neq 0$.

(49) If $a > 0$, then $a^{k}_{\mathbb{Z}} > 0$.

(50) $(a \cdot b)^{k}_{\mathbb{Z}} = (a^{k}_{\mathbb{Z}}) \cdot b^{k}_{\mathbb{Z}}$.

(51) If $a \neq 0$, then $a^{-k}_{\mathbb{Z}} = \frac{1}{a^{k}_{\mathbb{Z}}}$.

(52) $(\frac{1}{a})^{k}_{\mathbb{Z}} = \frac{1}{a^{k}_{\mathbb{Z}}}$.

(53) If $a \neq 0$, then $a^{m-n}_{\mathbb{Z}} = \frac{a^{m}}{a^{n}}$.

(54) If $a \neq 0$, then $a^{k+l}_{\mathbb{Z}} = (a^{k}_{\mathbb{Z}}) \cdot a^{l}_{\mathbb{Z}}$.

(55) $(a^{k}_{\mathbb{Z}})^{l}_{\mathbb{Z}} = a^{k \cdot l}_{\mathbb{Z}}$.

(56) If $a > 0$ and $n \geq 1$, then $(\mathrm{root}_{n}(a))^{k}_{\mathbb{Z}} = \mathrm{root}_{n}(a^{k}_{\mathbb{Z}})$.

Let a be a real number and let us consider p. The functor $a^{p}_{\mathbb{Q}}$ is defined by:

(Def. 5) $a^{p}_{\mathbb{Q}} = \mathrm{root}_{\mathrm{den}\, p}(a^{\mathrm{num}\, p}_{\mathbb{Z}})$.

Let a be a real number and let us consider p. Observe that $a^{p}_{\mathbb{Q}}$ is real.
Let a be a real number and let us consider p. Then $a^{p}_{\mathbb{Q}}$ is a real number.
The following propositions are true:

(58)[7] If $a > 0$ and $p = 0$, then $a^{p}_{\mathbb{Q}} = 1$.

(59) If $a > 0$ and $p = 1$, then $a^{p}_{\mathbb{Q}} = a$.

(60) If $a > 0$ and $p = n$, then $a^{p}_{\mathbb{Q}} = a^{n}$.

(61) If $a > 0$ and $n \geq 1$ and $p = n^{-1}$, then $a^{p}_{\mathbb{Q}} = \mathrm{root}_{n}(a)$.

(62) $1^{p}_{\mathbb{Q}} = 1$.

(63) If $a > 0$, then $a^{p}_{\mathbb{Q}} > 0$.

(64) If $a > 0$, then $(a^{p}_{\mathbb{Q}}) \cdot a^{q}_{\mathbb{Q}} = a^{p+q}_{\mathbb{Q}}$.

(65) If $a > 0$, then $\frac{1}{a^{p}_{\mathbb{Q}}} = a^{-p}_{\mathbb{Q}}$.

(66) If $a > 0$, then $\frac{a^{p}_{\mathbb{Q}}}{a^{q}_{\mathbb{Q}}} = a^{p-q}_{\mathbb{Q}}$.

[6] The proposition (43) has been removed.
[7] The proposition (57) has been removed.

(67) If $a > 0$ and $b > 0$, then $(a \cdot b)^p_{\mathbb{Q}} = (a^p_{\mathbb{Q}}) \cdot b^p_{\mathbb{Q}}$.

(68) If $a > 0$, then $(\frac{1}{a})^p_{\mathbb{Q}} = \frac{1}{a^p_{\mathbb{Q}}}$.

(69) If $a > 0$ and $b > 0$, then $(\frac{a}{b})^p_{\mathbb{Q}} = \frac{a^p_{\mathbb{Q}}}{b^p_{\mathbb{Q}}}$.

(70) If $a > 0$, then $(a^p_{\mathbb{Q}})^q_{\mathbb{Q}} = a^{p \cdot q}_{\mathbb{Q}}$.

(71) If $a \geq 1$ and $p \geq 0$, then $a^p_{\mathbb{Q}} \geq 1$.

(72) If $a \geq 1$ and $p \leq 0$, then $a^p_{\mathbb{Q}} \leq 1$.

(73) If $a > 1$ and $p > 0$, then $a^p_{\mathbb{Q}} > 1$.

(74) If $a \geq 1$ and $p \geq q$, then $a^p_{\mathbb{Q}} \geq a^q_{\mathbb{Q}}$.

(75) If $a > 1$ and $p > q$, then $a^p_{\mathbb{Q}} > a^q_{\mathbb{Q}}$.

(76) If $a > 0$ and $a < 1$ and $p > 0$, then $a^p_{\mathbb{Q}} < 1$.

(77) If $a > 0$ and $a \leq 1$ and $p \leq 0$, then $a^p_{\mathbb{Q}} \geq 1$.

Let I_1 be a sequence of real numbers. We say that I_1 is rational sequence-like if and only if:

(Def. 6) For every n holds $I_1(n)$ is a rational number.

Let us observe that there exists a sequence of real numbers which is rational sequence-like.
A rational sequence is a rational sequence-like sequence of real numbers.
Let s be a rational sequence and let us consider n. Then $s(n)$ is a rational number.
One can prove the following propositions:

(79)[8] For every real number a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \leq a$.

(80) For every a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \geq a$.

Let a be a real number and let s be a rational sequence. The functor $a^s_{\mathbb{Q}}$ yielding a sequence of real numbers is defined by:

(Def. 7) For every n holds $(a^s_{\mathbb{Q}})(n) = a^{s(n)}_{\mathbb{Q}}$.

Next we state two propositions:

(82)[9] For every rational sequence s such that s is convergent and $a > 0$ holds $a^s_{\mathbb{Q}}$ is convergent.

(83) Let s_1, s_2 be rational sequences and given a. Suppose s_1 is convergent and s_2 is convergent and $\lim s_1 = \lim s_2$ and $a > 0$. Then $a^{s_1}_{\mathbb{Q}}$ is convergent and $a^{s_2}_{\mathbb{Q}}$ is convergent and $\lim(a^{s_1}_{\mathbb{Q}}) = \lim(a^{s_2}_{\mathbb{Q}})$.

Let a, b be real numbers. Let us assume that $a > 0$. The functor $a^b_{\mathbb{R}}$ yields a real number and is defined as follows:

(Def. 8) There exists a rational sequence s such that s is convergent and $\lim s = b$ and $a^s_{\mathbb{Q}}$ is convergent and $\lim(a^s_{\mathbb{Q}}) = a^b_{\mathbb{R}}$.

Let a, b be real numbers. Then $a^b_{\mathbb{R}}$ is a real number.
Next we state a number of propositions:

[8] The proposition (78) has been removed.
[9] The proposition (81) has been removed.

(85)[10] If $a > 0$, then $a_{\mathbb{R}}^{0} = 1$.

(86) If $a > 0$, then $a_{\mathbb{R}}^{1} = a$.

(87) $1_{\mathbb{R}}^{a} = 1$.

(88) If $a > 0$, then $a_{\mathbb{R}}^{p} = a_{\mathbb{Q}}^{p}$.

(89) If $a > 0$, then $a_{\mathbb{R}}^{b+c} = (a_{\mathbb{R}}^{b}) \cdot a_{\mathbb{R}}^{c}$.

(90) If $a > 0$, then $a_{\mathbb{R}}^{-c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(91) If $a > 0$, then $a_{\mathbb{R}}^{b-c} = \frac{a_{\mathbb{R}}^{b}}{a_{\mathbb{R}}^{c}}$.

(92) If $a > 0$ and $b > 0$, then $(a \cdot b)_{\mathbb{R}}^{c} = (a_{\mathbb{R}}^{c}) \cdot b_{\mathbb{R}}^{c}$.

(93) If $a > 0$, then $(\frac{1}{a})_{\mathbb{R}}^{c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(94) If $a > 0$ and $b > 0$, then $(\frac{a}{b})_{\mathbb{R}}^{c} = \frac{a_{\mathbb{R}}^{c}}{b_{\mathbb{R}}^{c}}$.

(95) If $a > 0$, then $a_{\mathbb{R}}^{b} > 0$.

(96) If $a \geq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \geq a_{\mathbb{R}}^{b}$.

(97) If $a > 1$ and $c > b$, then $a_{\mathbb{R}}^{c} > a_{\mathbb{R}}^{b}$.

(98) If $a > 0$ and $a \leq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \leq a_{\mathbb{R}}^{b}$.

(99) If $a \geq 1$ and $b \geq 0$, then $a_{\mathbb{R}}^{b} \geq 1$.

(100) If $a > 1$ and $b > 0$, then $a_{\mathbb{R}}^{b} > 1$.

(101) If $a \geq 1$ and $b \leq 0$, then $a_{\mathbb{R}}^{b} \leq 1$.

(102) If $a > 1$ and $b < 0$, then $a_{\mathbb{R}}^{b} < 1$.

(103) If s_1 is convergent and s_2 is convergent and $\lim s_1 > 0$ and for every n holds $s_1(n) > 0$ and $s_2(n) = s_1(n)_{\mathbb{Q}}^{p}$, then $\lim s_2 = (\lim s_1)_{\mathbb{Q}}^{p}$.

(104) If $a > 0$ and s_1 is convergent and s_2 is convergent and for every n holds $s_2(n) = a_{\mathbb{R}}^{s_1(n)}$, then $\lim s_2 = a_{\mathbb{R}}^{\lim s_1}$.

(105) If $a > 0$, then $(a_{\mathbb{R}}^{b})_{\mathbb{R}}^{c} = a_{\mathbb{R}}^{b \cdot c}$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[5] Andrzej Kondracki. Basic properties of rational numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rat_1.html`.

[6] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_2.html`.

[10] The proposition (84) has been removed.

[7] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[8] Rafał Kwiatek. Factorial and Newton coefficients. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/newton.html.

[9] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[10] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[11] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[12] Michał J. Trybulec. Integers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/int_1.html.

[13] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/group_1.html.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

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