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Preliminaries to the Lambek Calculus

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Summary. Some preliminary facts concerning completeness and decidability problems for the Lambek Calculus [14] are proved as well as some theses and derived rules of the calculus itself.

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The articles [16], [9], [19], [18], [13], [1], [6], [21], [5], [10], [12], [17], [20], [7], [8], [15], [11], [2], [3], and [4] provide the notation and terminology for this paper.

We consider structures of the type algebra as extensions of 1-sorted structure as systems
⟨ a carrier, a left quotient, a right quotient, an inner product ⟩,
where the carrier is a set and the left quotient, the right quotient, and the inner product are binary operations on the carrier.

One can check that there exists a structure of the type algebra which is non empty and strict.

Let s be a non empty structure of the type algebra. A type of s is an element of s.

We follow the rules: s denotes a non empty structure of the type algebra, T, X, Y denote finite sequences of elements of the carrier of s, and x, y, z denote types of s.

Let us consider s, x, y. The functor $x \setminus y$ yields a type of s and is defined by:

(Def. 1) $x \setminus y = (\text{the left quotient of } s)(x, y)$.

The functor x/y yields a type of s and is defined as follows:

(Def. 2) $x/y = (\text{the right quotient of } s)(x, y)$.

The functor $x \cdot y$ yielding a type of s is defined as follows:

(Def. 3) $x \cdot y = (\text{the inner product of } s)(x, y)$.

Let T_1 be a finite tree and let v be an element of T_1. One can check that $\operatorname{succ} v$ is finite.

Let T_1 be a finite tree and let v be an element of T_1. The branch degree of v is defined as follows:

(Def. 4) The branch degree of $v = \operatorname{card} \operatorname{succ} v$.

Let us note that there exists a decorated tree which is finite.

Let D be a non empty set. One can verify that there exists a tree decorated with elements of D which is finite.

Let us consider s. A preproof of s is a finite tree decorated with elements of $[: [: (\text{the carrier of } s)^*, \text{the carrier of } s :], \mathbb{N} :]$.

In the sequel T_1 is a preproof of s.

Let R be a finite binary relation. One can verify that $\operatorname{dom} R$ is finite.

Let us consider s, T_1 and let v be an element of $\operatorname{dom} T_1$. We say that v is correct if and only if:

(Def. 5)(i) The branch degree of $v = 0$ and there exists x such that $T_1(v)_{\mathbf{1}} = \langle\langle x\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 0$,

(ii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, x/y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T \frown \langle y\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 1$,

(iii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, y \setminus x\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle\langle y\rangle \frown T, x\rangle$ if $T_1(v)_{\mathbf{2}} = 2$,

(iv) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x/y\rangle \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 3$,

(v) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 4$,

(vi) the branch degree of $v = 1$ and there exist X, x, y, Y such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 5$,

(vii) the branch degree of $v = 2$ and there exist X, Y, x, y such that $T_1(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X, x\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle Y, y\rangle$ if $T_1(v)_{\mathbf{2}} = 6$,

(viii) the branch degree of $v = 2$ and there exist T, X, Y, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 7$,

(ix) *contradiction*,[1] otherwise.

Let us consider s and let I_1 be a type of s. We say that I_1 is left if and only if:

(Def. 6) There exist x, y such that $I_1 = x \setminus y$.

We say that I_1 is right if and only if:

(Def. 7) There exist x, y such that $I_1 = x/y$.

We say that I_1 is middle if and only if:

(Def. 8) There exist x, y such that $I_1 = x \cdot y$.

Let us consider s and let I_1 be a type of s. We say that I_1 is primitive if and only if:

(Def. 9) I_1 is left, right, and middle.

Let us consider s, let T_1 be a finite tree decorated with elements of the carrier of s, and let v be an element of $\operatorname{dom} T_1$. Then $T_1(v)$ is a type of s.

Let us consider s, let T_1 be a finite tree decorated with elements of the carrier of s, and let us consider x. We say that T_1 represents x if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $\operatorname{dom} T_1$ is finite, and

(ii) for every element v of $\operatorname{dom} T_1$ holds the branch degree of $v = 0$ or the branch degree of $v = 2$ but if the branch degree of $v = 0$, then $T_1(v)$ is primitive but if the branch degree of $v = 2$, then there exist y, z such that $T_1(v) = y/z$ or $T_1(v) = y \setminus z$ or $T_1(v) = y \cdot z$ but $T_1(v \frown \langle 0\rangle) = y$ but $T_1(v \frown \langle 1\rangle) = z$.

We introduce T_1 does not represent x as an antonym of T_1 represents x.

Let I_1 be a non empty structure of the type algebra. We say that I_1 is free if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) There exists no type of I_1 which is left, right, left, middle, right, and middle, and

(ii) for every type x of I_1 there exists a finite tree T_1 decorated with elements of the carrier of I_1 such that for every finite tree T_2 decorated with elements of the carrier of I_1 holds T_2 represents x iff $T_1 = T_2$.

Let us consider s, x. Let us assume that s is free. The representation of x yielding a finite tree decorated with elements of the carrier of s is defined by the condition (Def. 12).

[1] This definition is absolutely permissive, i.e. we assume a *contradiction*, but we are interested only in the type of the functor 'choose'.

(Def. 12) Let T_1 be a finite tree decorated with elements of the carrier of s. Then T_1 represents x if and only if the representation of $x = T_1$.

Let us consider s, let f be a finite sequence of elements of the carrier of s, and let t be a type of s. Then $\langle f, t\rangle$ is an element of [:(the carrier of s)*, the carrier of s:].

Let us consider s. A preproof of s is called a proof of s if:

(Def. 13) dom it is finite and every element of dom it is correct.

In the sequel p denotes a proof of s and v denotes an element of dom p.

One can prove the following propositions:

(1) If the branch degree of $v = 1$, then $v \frown \langle 0\rangle \in \operatorname{dom} p$.

(2) If the branch degree of $v = 2$, then $v \frown \langle 0\rangle \in \operatorname{dom} p$ and $v \frown \langle 1\rangle \in \operatorname{dom} p$.

(3) If $p(v)_{\mathbf{2}} = 0$, then there exists x such that $p(v)_{\mathbf{1}} = \langle\langle x\rangle, x\rangle$.

(4) If $p(v)_{\mathbf{2}} = 1$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0\rangle$ and $p(v)_{\mathbf{1}} = \langle T, x/y\rangle$ and $p(w)_{\mathbf{1}} = \langle T \frown \langle y\rangle, x\rangle$.

(5) If $p(v)_{\mathbf{2}} = 2$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0\rangle$ and $p(v)_{\mathbf{1}} = \langle T, y \setminus x\rangle$ and $p(w)_{\mathbf{1}} = \langle\langle y\rangle \frown T, x\rangle$.

(6) Suppose $p(v)_{\mathbf{2}} = 3$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x/y\rangle \frown T \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle T, y\rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$.

(7) Suppose $p(v)_{\mathbf{2}} = 4$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x\rangle \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle T, y\rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$.

(8) Suppose $p(v)_{\mathbf{2}} = 5$. Then there exists an element w of dom p and there exist X, x, y, Y such that $w = v \frown \langle 0\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y\rangle \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown \langle y\rangle \frown Y, z\rangle$.

(9) Suppose $p(v)_{\mathbf{2}} = 6$. Then there exist elements w, u of dom p and there exist X, Y, x, y such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y\rangle$ and $p(w)_{\mathbf{1}} = \langle X, x\rangle$ and $p(u)_{\mathbf{1}} = \langle Y, y\rangle$.

(10) Suppose $p(v)_{\mathbf{2}} = 7$. Then there exist elements w, u of dom p and there exist T, X, Y, y, z such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle T, y\rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$.

(11) $p(v)_{\mathbf{2}} = 0$ or $p(v)_{\mathbf{2}} = 1$ or $p(v)_{\mathbf{2}} = 2$ or $p(v)_{\mathbf{2}} = 3$ or $p(v)_{\mathbf{2}} = 4$ or $p(v)_{\mathbf{2}} = 5$ or $p(v)_{\mathbf{2}} = 6$ or $p(v)_{\mathbf{2}} = 7$.

Let us consider s and let I_1 be a preproof of s. We say that I_1 is cut-free if and only if:

(Def. 14) For every element v of dom I_1 holds $I_1(v)_{\mathbf{2}} \neq 7$.

Let us consider s. The size w.r.t. s yielding a function from the carrier of s into $\mathbb{N}$ is defined by:

(Def. 15) For every x holds (the size w.r.t. s)(x) = card dom(the representation of x).

Let D be a non empty set, let T be a finite sequence of elements of D, and let f be a function from D into $\mathbb{N}$. Then $f \cdot T$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider s and let p be a proof of s. The cut degree of p yielding a natural number is defined by:

(Def. 16)(i) There exist T, X, Y, y, z such that $p(\emptyset)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $p(\langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $p(\langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ and the cut degree of p = (the size w.r.t. s)(y) + (the size w.r.t. s)(z) + $\sum$((the size w.r.t. s) $\cdot$ ($X \frown T \frown Y$)) if $p(\emptyset)_{\mathbf{2}} = 7$,

(ii) the cut degree of $p = 0$, otherwise.

We use the following convention: A is a non empty set and a, a_1, a_2, b are elements of A^*.

Let us consider s, A. A function from the carrier of s into 2^{A^*} is said to be a model of s if it satisfies the condition (Def. 17).

(Def. 17) Let given x, y. Then

(i) $\mathrm{it}(x \cdot y) = \{a \frown b : a \in \mathrm{it}(x) \wedge b \in \mathrm{it}(y)\}$,

(ii) $\mathrm{it}(x/y) = \{a_1 : \bigwedge_b (b \in \mathrm{it}(y) \Rightarrow a_1 \frown b \in \mathrm{it}(x))\}$, and

(iii) $\mathrm{it}(y \setminus x) = \{a_2 : \bigwedge_b (b \in \mathrm{it}(y) \Rightarrow b \frown a_2 \in \mathrm{it}(x))\}$.

Let a, b be non empty sets. Observe that there exists a relation between a and b which is non empty.

We consider type structures as extensions of structure of the type algebra as systems

⟨ a carrier, a left quotient, a right quotient, an inner product, a derivability ⟩,

where the carrier is a set, the left quotient, the right quotient, and the inner product are binary operations on the carrier, and the derivability is a relation between the carrier* and the carrier.

Let us observe that there exists a type structure which is non empty and strict.

In the sequel s is a non empty type structure and x is a type of s.

Let us consider s, let f be a finite sequence of elements of the carrier of s, and let us consider x. The predicate $f \longrightarrow x$ is defined by:

(Def. 18) $\langle f, x \rangle \in$ the derivability of s.

Let I_1 be a non empty type structure. We say that I_1 is calculus of syntactic types-like if and only if the conditions (Def. 19) are satisfied.

(Def. 19) For every type x of I_1 holds $\langle x \rangle \longrightarrow x$ and for every finite sequence T of elements of the carrier of I_1 and for all types x, y of I_1 such that $T \frown \langle y \rangle \longrightarrow x$ holds $T \longrightarrow x/y$ and for every finite sequence T of elements of the carrier of I_1 and for all types x, y of I_1 such that $\langle y \rangle \frown T \longrightarrow x$ holds $T \longrightarrow y \setminus x$ and for all finite sequences T, X, Y of elements of the carrier of I_1 and for all types x, y, z of I_1 such that $T \longrightarrow y$ and $X \frown \langle x \rangle \frown Y \longrightarrow z$ holds $X \frown \langle x/y \rangle \frown T \frown Y \longrightarrow z$ and for all finite sequences T, X, Y of elements of the carrier of I_1 and for all types x, y, z of I_1 such that $T \longrightarrow y$ and $X \frown \langle x \rangle \frown Y \longrightarrow z$ holds $X \frown T \frown \langle y \setminus x \rangle \frown Y \longrightarrow z$ and for all finite sequences X, Y of elements of the carrier of I_1 and for all types x, y, z of I_1 such that $X \frown \langle x \rangle \frown \langle y \rangle \frown Y \longrightarrow z$ holds $X \frown \langle x \cdot y \rangle \frown Y \longrightarrow z$ and for all finite sequences X, Y of elements of the carrier of I_1 and for all types x, y of I_1 such that $X \longrightarrow x$ and $Y \longrightarrow y$ holds $X \frown Y \longrightarrow x \cdot y$.

Let us observe that there exists a non empty type structure which is calculus of syntactic types-like.

A calculus of syntactic types is a calculus of syntactic types-like non empty type structure.

In the sequel s denotes a calculus of syntactic types and x, y, z denote types of s.

Next we state a number of propositions:

(12) $\langle x/y \rangle \frown \langle y \rangle \longrightarrow x$ and $\langle y \rangle \frown \langle y \setminus x \rangle \longrightarrow x$.

(13) $\langle x \rangle \longrightarrow y/(x \setminus y)$ and $\langle x \rangle \longrightarrow y/x \setminus y$.

(14) $\langle x/y \rangle \longrightarrow x/z/(y/z)$.

(15) $\langle y \setminus x \rangle \longrightarrow z \setminus y \setminus (z \setminus x)$.

(16) If $\langle x \rangle \longrightarrow y$, then $\langle x/z \rangle \longrightarrow y/z$ and $\langle z \setminus x \rangle \longrightarrow z \setminus y$.

(17) If $\langle x \rangle \longrightarrow y$, then $\langle z/y \rangle \longrightarrow z/x$ and $\langle y \setminus z \rangle \longrightarrow x \setminus z$.

(18) $\langle y/(y/x \setminus y) \rangle \longrightarrow y/x$.

(19) If $\langle x \rangle \longrightarrow y$, then $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y/x$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \setminus y$.

(20) $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x/x$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \setminus x$.

(21) $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y/(x \setminus y)/x$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \setminus (y/x \setminus y)$.

(22) $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x/z/(y/z)/(x/y)$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y \setminus x \setminus (z \setminus y \setminus (z \setminus x))$.

(23) If $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x$, then $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y/(y/x)$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \setminus y \setminus y$.

(24) $\langle x/(y/y) \rangle \longrightarrow x$.

Let us consider s, x, y. The predicate $x \longleftrightarrow y$ is defined as follows:

(Def. 20) $\langle x \rangle \longrightarrow y$ and $\langle y \rangle \longrightarrow x$.

The following propositions are true:

(25) $x \longleftrightarrow x$.

(26) $x/y \longleftrightarrow x/(x/y \setminus x)$.

(27) $x/(z \cdot y) \longleftrightarrow x/y/z$.

(28) $\langle x \cdot (y/z) \rangle \longrightarrow (x \cdot y)/z$.

(29) $\langle x \rangle \longrightarrow (x \cdot y)/y$ and $\langle x \rangle \longrightarrow y \setminus y \cdot x$.

(30) $(x \cdot y) \cdot z \longleftrightarrow x \cdot (y \cdot z)$.

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