JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Preliminaries to Circuits, I[1]

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Summary. This article is the first in a series of four articles (continued in [23],[22],[24]) about modelling circuits by many-sorted algebras.

Here, we introduce some auxiliary notations and prove auxiliary facts about many sorted sets, many sorted functions and trees.

MML Identifier: `PRE_CIRC`.

WWW: `http://mizar.org/JFM/Vol6/pre_circ.html`

The articles [26], [15], [31], [4], [30], [2], [1], [5], [29], [19], [32], [13], [18], [14], [25], [17], [7], [3], [9], [10], [11], [6], [8], [27], [20], [28], [21], [12], and [16] provide the notation and terminology for this paper.

1. Varia

The scheme *FraenkelFinIm* deals with a finite non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x); x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\}$ is finite

for all values of the parameters.

Next we state three propositions:

(2)[1] For every function f and for all sets x, y such that $\mathrm{dom} f = \{x\}$ and $\mathrm{rng} f = \{y\}$ holds $f = \{\langle x, y\rangle\}$.

(3) For all functions f, g, h such that $f \subseteq g$ holds $f {+\cdot} h \subseteq g {+\cdot} h$.

(4) For all functions f, g, h such that $f \subseteq g$ and $\mathrm{dom} f$ misses $\mathrm{dom} h$ holds $f \subseteq g {+\cdot} h$.

Let us note that there exists a set which is finite, non empty, and natural-membered.

Let A be a finite non empty real-membered set. Then $\sup A$ is a real number and it can be characterized by the condition:

(Def. 1) $\sup A \in A$ and for every real number k such that $k \in A$ holds $k \leq \sup A$.

We introduce $\max A$ as a synonym of $\sup A$.

Let X be a finite non empty natural-membered set. One can verify that $\max X$ is natural.

[1]This work was initiated while the second author visited Nagano (March–May 1994) and then continued when the third author visited Edmonton (May–June 1994). The work was finalized when the fourth author visited Białystok (October–November 1994). Partial funding for this work has been provided by: Shinshu Endowment Fund for Information Science, NSERC Grant OGP9207, JSTF award 651-93-S009.

[1] The proposition (1) has been removed.

2. Many Sorted Sets and Functions

The following proposition is true

(5) For every set I and for every many sorted set M_1 indexed by I holds ${M_1}^{\#}(\varepsilon_I) = \{\emptyset\}$.

The scheme *MSSLambda2Part* deals with a set $\mathcal{A}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a unary predicate $\mathcal{P}$, and states that:

There exists a many sorted set f indexed by $\mathcal{A}$ such that for every element i of $\mathcal{A}$ holds

(i) if $\mathcal{P}[i]$, then $f(i) = \mathcal{F}(i)$, and

(ii) if not $\mathcal{P}[i]$, then $f(i) = \mathcal{G}(i)$

for all values of the parameters.

Let I be a set and let I_1 be a many sorted set indexed by I. We say that I_1 is locally-finite if and only if:

(Def. 3)[2] For every set i such that $i \in I$ holds $I_1(i)$ is finite.

Let I be a set. One can verify that there exists a many sorted set indexed by I which is non-empty and locally-finite.

Let I, A be sets. Then $I \longmapsto A$ is a many sorted set indexed by I.

Let I be a set, let M be a many sorted set indexed by I, and let A be a subset of I. Then $M{\upharpoonright}A$ is a many sorted set indexed by A.

Let M be a non-empty function and let A be a set. Observe that $M{\upharpoonright}A$ is non-empty.

Next we state three propositions:

(6) For every non empty set I and for every non-empty many sorted set B indexed by I holds $\bigcup \mathrm{rng}\, B$ is non empty.

(7) For every set I holds $\mathrm{uncurry}(I \longmapsto \emptyset) = \emptyset$.

(8) Let I be a non empty set, A be a set, B be a non-empty many sorted set indexed by I, and F be a many sorted function from $I \longmapsto A$ into B. Then $\mathrm{dom}\,\mathrm{commute}(F) = A$.

Now we present two schemes. The scheme *LambdaRecCorrD* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

(i) There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every natural number i holds $f(i+1) = \mathcal{F}(i, f(i))$, and

(ii) for all functions f_1, f_2 from $\mathbb{N}$ into $\mathcal{A}$ such that $f_1(0) = \mathcal{B}$ and for every natural number i holds $f_1(i+1) = \mathcal{F}(i, f_1(i))$ and $f_2(0) = \mathcal{B}$ and for every natural number i holds $f_2(i+1) = \mathcal{F}(i, f_2(i))$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *LambdaMSFD* deals with a non empty set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, many sorted sets $\mathcal{C}$, $\mathcal{D}$ indexed by $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a many sorted function f from $\mathcal{C}$ into $\mathcal{D}$ such that for every element i of $\mathcal{A}$ such that $i \in \mathcal{B}$ holds $f(i) = \mathcal{F}(i)$

provided the parameters have the following property:

- For every element i of $\mathcal{A}$ such that $i \in \mathcal{B}$ holds $\mathcal{F}(i)$ is a function from $\mathcal{C}(i)$ into $\mathcal{D}(i)$.

Let F be a non-empty function and let f be a function. Observe that $F \cdot f$ is non-empty.

Let I be a set and let M_1 be a non-empty many sorted set indexed by I. One can verify that every element of $\prod M_1$ is function-like and relation-like.

Next we state four propositions:

(9) Let I be a set, f be a non-empty many sorted set indexed by I, g be a function, and s be an element of $\prod f$. Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, g$ holds $g(x) \in f(x)$. Then $s {+\cdot} g$ is an element of $\prod f$.

[2] The definition (Def. 2) has been removed.

(10) Let A, B be non empty sets, C be a non-empty many sorted set indexed by A, I_2 be a many sorted function from $A \longmapsto B$ into C, and b be an element of B. Then there exists a many sorted set c indexed by A such that $c = (\mathrm{commute}(I_2))(b)$ and $c \in C$.

(11) Let I be a set, M be a many sorted set indexed by I, and x, g be functions. If $x \in \prod M$, then $x \cdot g \in \prod(M \cdot g)$.

(12) For every natural number n and for every set a holds $\prod(n \mapsto \{a\}) = \{n \mapsto a\}$.

3. TREES

We adopt the following rules: T, T_1 denote finite trees, t, p denote elements of T, and t_1 denotes an element of T_1.

Let D be a non empty set. Observe that every element of $\mathrm{FinTrees}(D)$ is finite.

Let T be a finite decorated tree and let t be an element of $\mathrm{dom}\, T$. One can check that $T \upharpoonright t$ is finite.

One can prove the following proposition

(13) $T \upharpoonright p \approx \{t : p \preceq t\}$.

Let T be a finite decorated tree, let t be an element of $\mathrm{dom}\, T$, and let T_1 be a finite decorated tree. Observe that T with-replacement(t, T_1) is finite.

Next we state a number of propositions:

(14) T with-replacement$(p, T_1) = \{t : p \not\preceq t\} \cup \{p \frown t_1\}$.

(15) For every finite sequence f of elements of $\mathbb{N}$ such that $f \in T$ with-replacement(p, T_1) and $p \preceq f$ there exists t_1 such that $f = p \frown t_1$.

(16) For every tree yielding finite sequence p and for every natural number k such that $k+1 \in \mathrm{dom}\, p$ holds $\overbrace{p} \upharpoonright \langle k \rangle = p(k+1)$.

(17) Let q be a decorated tree yielding finite sequence and k be a natural number. If $k+1 \in \mathrm{dom}\, q$, then $\langle k \rangle \in \overbrace{\mathop{\mathrm{dom}}_{\kappa} q(\kappa)}$.

(18) Let p, q be tree yielding finite sequences and k be a natural number. Suppose $\mathrm{len}\, p = \mathrm{len}\, q$ and $k+1 \in \mathrm{dom}\, p$ and for every natural number i such that $i \in \mathrm{dom}\, p$ and $i \neq k+1$ holds $p(i) = q(i)$. Let t be a tree. If $q(k+1) = t$, then $\overbrace{q} = \overbrace{p}$ with-replacement$(\langle k \rangle, t)$.

(19) Let e_1, e_2 be finite decorated trees, x be a set, k be a natural number, and p be a decorated tree yielding finite sequence. Suppose $\langle k \rangle \in \mathrm{dom}\, e_1$ and $e_1 = x$-tree(p). Then there exists a decorated tree yielding finite sequence q such that e_1 with-replacement$(\langle k \rangle, e_2) = x$-tree$(q)$ and $\mathrm{len}\, q = \mathrm{len}\, p$ and $q(k+1) = e_2$ and for every natural number i such that $i \in \mathrm{dom}\, p$ and $i \neq k+1$ holds $q(i) = p(i)$.

(20) For every finite tree T and for every element p of T such that $p \neq \emptyset$ holds $\mathrm{card}(T \upharpoonright p) < \mathrm{card}\, T$.

(21) For every function f holds $\overline{\overline{(f\ \textbf{qua}\ \mathrm{set})}} = \overline{\overline{\mathrm{dom}\, f}}$.

(22) For all finite trees T, T_1 and for every element p of T holds $\mathrm{card}(T$ with-replacement$(p, T_1)) + \mathrm{card}(T \upharpoonright p) = \mathrm{card}\, T + \mathrm{card}\, T_1$.

(23) For all finite decorated trees T, T_1 and for every element p of $\mathrm{dom}\, T$ holds $\mathrm{card}(T$ with-replacement$(p, T_1)) + \mathrm{card}(T \upharpoonright p) = \mathrm{card}\, T + \mathrm{card}\, T_1$.

Let x be a set. One can check that the root tree of x is finite.

One can prove the following proposition

(24) For every set x holds card(the root tree of x) $= 1$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/trees_1.html.

[4] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[5] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal2.html.

[6] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_5.html.

[7] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[8] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/funct_6.html.

[9] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/trees_2.html.

[10] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/trees_3.html.

[11] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/trees_4.html.

[12] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[13] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[14] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[15] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[16] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[17] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[18] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[19] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_4.html.

[20] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[21] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pralg_2.html.

[22] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/circuit1.html.

[23] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree2.html.

[24] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, II. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/circuit2.html.

[25] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[26] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[27] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[28] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[29] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol16/membered.html.

[30] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[31] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[32] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received November 17, 1994

Published January 2, 2004