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More on Products of Many Sorted Algebras

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Summary. This article is continuation of an article defining products of many sorted algebras [11]. Some properties of notions such as commute, Frege, Args() are shown in this article. Notions of constant of operations in many sorted algebras and projection of products of family of many sorted algebras are defined. There is also introduced the notion of class of family of many sorted algebras. The main theorem states that product of family of many sorted algebras and product of class of family of many sorted algebras are isomorphic.

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The articles [14], [19], [20], [6], [21], [7], [15], [8], [13], [4], [1], [3], [2], [16], [17], [9], [11], [12], [18], [10], and [5] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: I is a non empty set, J is a many sorted set indexed by I, S is a non void non empty many sorted signature, i is an element of I, c is a set, A is an algebra family of I over S, E_1 is an equivalence relation of I, U_0, U_1, U_2 are algebras over S, s is a sort symbol of S, o is an operation symbol of S, and f is a function.

Let I be a set, let us consider S, and let A_1 be an algebra family of I over S. Observe that $\prod A_1$ is non-empty.

Let I be a set. Then id_I is a many sorted set indexed by I.

Let us consider I, E_1. Observe that $\mathrm{Classes}\,E_1$ has non empty elements.

Let X be a set with non empty elements. Then id_X is a non-empty many sorted set indexed by X.

One can prove the following propositions:

(1) For all functions f, F and for every set A such that $f \in \prod F$ holds $f \restriction A \in \prod (F \restriction A)$.

(2) Let A be an algebra family of I over S, s be a sort symbol of S, a be a non empty subset of I, and A_2 be an algebra family of a over S. If $A \restriction a = A_2$, then $\mathrm{Carrier}(A_2, s) = \mathrm{Carrier}(A, s) \restriction a$.

(3) Let i be a set, I be a non empty set, E_1 be an equivalence relation of I, and c_1, c_2 be elements of $\mathrm{Classes}\,E_1$. If $i \in c_1$ and $i \in c_2$, then $c_1 = c_2$.

(4) For all sets X, Y and for every function f such that $f \in Y^X$ holds $\mathrm{dom}\, f = X$ and $\mathrm{rng}\, f \subseteq Y$.

(5) Let D be a non empty set, F be a many sorted function indexed by D, and C be a functional non empty set with common domain. Suppose $C = \mathrm{rng}\, F$. Let d be an element of D and e be a set. If $d \in \mathrm{dom}\, F$ and $e \in \mathrm{DOM}(C)$, then $F(d)(e) = (\mathrm{commute}(F))(e)(d)$.

2. Constants of Many Sorted Algebras

Let us consider S, U_0 and let o be an operation symbol of S. The functor $\mathrm{const}(o,U_0)$ is defined as follows:

(Def. 1) $\mathrm{const}(o,U_0) = (\mathrm{Den}(o,U_0))(\emptyset)$.

We now state four propositions:

(6) If $\mathrm{Arity}(o) = \emptyset$ and $\mathrm{Result}(o,U_0) \neq \emptyset$, then $\mathrm{const}(o,U_0) \in \mathrm{Result}(o,U_0)$.

(7) Suppose (the sorts of $U_0)(s) \neq \emptyset$. Then $\mathrm{Constants}(U_0,s) = \{\mathrm{const}(o,U_0); o$ ranges over elements of the operation symbols of S: the result sort of $o = s \wedge \mathrm{Arity}(o) = \emptyset\}$.

(8) If $\mathrm{Arity}(o) = \emptyset$, then $(\mathrm{commute}(\mathrm{OPER}(A)))(o) \in ((\bigcup\{\mathrm{Result}(o,A(i')) : i'$ ranges over elements of $I\})^{\{\emptyset\}})^I$.

(9) If $\mathrm{Arity}(o) = \emptyset$, then $\mathrm{const}(o,\prod A) \in (\bigcup\{\mathrm{Result}(o,A(i')) : i'$ ranges over elements of $I\})^I$.

Let us consider S, I, o, A. Note that $\mathrm{const}(o,\prod A)$ is relation-like and function-like.
We now state three propositions:

(10) For every operation symbol o of S such that $\mathrm{Arity}(o) = \emptyset$ holds $(\mathrm{const}(o,\prod A))(i) = \mathrm{const}(o,A(i))$.

(11) If $\mathrm{Arity}(o) = \emptyset$ and $\mathrm{dom} f = I$ and for every element i of I holds $f(i) = \mathrm{const}(o,A(i))$, then $f = \mathrm{const}(o,\prod A)$.

(12) Let e be an element of $\mathrm{Args}(o,U_1)$. Suppose $e = \emptyset$ and $\mathrm{Arity}(o) = \emptyset$ and $\mathrm{Args}(o,U_1) \neq \emptyset$ and $\mathrm{Args}(o,U_2) \neq \emptyset$. Let F be a many sorted function from U_1 into U_2. Then $F\#e = \emptyset$.

3. Properties of Arguments of Operations in Many Sorted Algebras

We now state a number of propositions:

(13) Let U_1, U_2 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and x be an element of $\mathrm{Args}(o,U_1)$. Then $x \in \prod(\mathrm{dom}_\kappa(F \cdot \mathrm{Arity}(o))(\kappa))$.

(14) Let U_1, U_2 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, x be an element of $\mathrm{Args}(o,U_1)$, and n be a set. If $n \in \mathrm{dom}\,\mathrm{Arity}(o)$, then $(F\#x)(n) = F(\mathrm{Arity}(o)_n)(x(n))$.

(15) Let x be an element of $\mathrm{Args}(o,\prod A)$. Then $x \in ((\bigcup\{(\text{the sorts of } A(i'))(s') : i'$ ranges over elements of I, s' ranges over elements of the carrier of $S\})^I)^{\mathrm{dom}\,\mathrm{Arity}(o)}$.

(16) For every element x of $\mathrm{Args}(o,\prod A)$ and for every set n such that $n \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $x(n) \in \prod \mathrm{Carrier}(A,\mathrm{Arity}(o)_n)$.

(17) Let i be an element of I and n be a set. Suppose $n \in \mathrm{dom}\,\mathrm{Arity}(o)$. Let s be a sort symbol of S. Suppose $s = \mathrm{Arity}(o)(n)$. Let y be an element of $\mathrm{Args}(o,\prod A)$ and g be a function. If $g = y(n)$, then $g(i) \in (\text{the sorts of } A(i))(s)$.

(18) For every element y of $\mathrm{Args}(o,\prod A)$ such that $\mathrm{Arity}(o) \neq \emptyset$ holds $\mathrm{commute}(y) \in \prod(\mathrm{dom}_\kappa A(o)(\kappa))$.

(19) For every element y of $\mathrm{Args}(o,\prod A)$ such that $\mathrm{Arity}(o) \neq \emptyset$ holds $y \in \mathrm{dom}\,\blacksquare\mathrm{commute}(\mathrm{Frege}(A(o)))$.

(20) Let I be a set, S be a non void non empty many sorted signature, A be an algebra family of I over S, o be an operation symbol of S, and x be an element of $\mathrm{Args}(o,\prod A)$. Then $(\mathrm{Den}(o, \prod A))(x) \in \prod \mathrm{Carrier}(A, \text{the result sort of } o)$.

(21) Let given I, S, A, i and o be an operation symbol of S. Suppose $\mathrm{Arity}(o) \neq \emptyset$. Let U_1 be a non-empty algebra over S and x be an element of $\mathrm{Args}(o, \prod A)$. Then $(\mathrm{commute}(x))(i)$ is an element of $\mathrm{Args}(o, A(i))$.

(22) Let given I, S, A, i, o, x be an element of $\mathrm{Args}(o, \prod A)$, and n be a set. If $n \in \mathrm{dom}\,\mathrm{Arity}(o)$, then for every function f such that $f = x(n)$ holds $(\mathrm{commute}(x))(i)(n) = f(i)$.

(23) Let o be an operation symbol of S. Suppose $\mathrm{Arity}(o) \neq \emptyset$. Let y be an element of $\mathrm{Args}(o, \prod A)$, i' be an element of I, and g be a function. If $g = (\mathrm{Den}(o, \prod A))(y)$, then $g(i') = (\mathrm{Den}(o, A(i')))((\mathrm{commute}(y))(i'))$.

4. The Projection of Family of Many Sorted Algebras

Let f be a function and let x be a set. The functor $\mathrm{proj}(f,x)$ yielding a function is defined as follows:

(Def. 2) $\mathrm{dom}\,\mathrm{proj}(f,x) = \prod f$ and for every function y such that $y \in \mathrm{dom}\,\mathrm{proj}(f,x)$ holds $(\mathrm{proj}(f,x))(y) = y(x)$.

Let us consider I, S, let A be an algebra family of I over S, and let i be an element of I. The functor $\mathrm{proj}(A,i)$ yielding a many sorted function from $\prod A$ into $A(i)$ is defined as follows:

(Def. 3) For every element s of S holds $(\mathrm{proj}(A,i))(s) = \mathrm{proj}(\mathrm{Carrier}(A,s), i)$.

One can prove the following propositions:

(24) For every element x of $\mathrm{Args}(o, \prod A)$ such that $\mathrm{Args}(o, \prod A) \neq \emptyset$ and $\mathrm{Arity}(o) \neq \emptyset$ and for every element i of I holds $\mathrm{proj}(A,i)\#x = (\mathrm{commute}(x))(i)$.

(25) For every element i of I and for every algebra family A of I over S holds $\mathrm{proj}(A,i)$ is a homomorphism of $\prod A$ into $A(i)$.

(26) Let U_1 be a non-empty algebra over S and F be a many sorted function indexed by I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$. Then $F \in (\{F(i')(s_1) : s_1$ ranges over sort symbols of S, i' ranges over elements of $I\}^{\text{the carrier of } S})^I$ and $(\mathrm{commute}(F))(s)(i) = F(i)(s)$.

(27) Let U_1 be a non-empty algebra over S and F be a many sorted function indexed by I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$. Then $(\mathrm{commute}(F))(s) \in ((\bigcup\{(\text{the sorts of } A(i'))(s_1) : i'$ ranges over elements of I, s_1 ranges over sort symbols of $S\})^{(\text{the sorts of } U_1)(s)})^I$.

(28) Let U_1 be a non-empty algebra over S and F be a many sorted function indexed by I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$. Let F' be a many sorted function from U_1 into $A(i)$. Suppose $F' = F(i)$. Let x be a set. Suppose $x \in (\text{the sorts of } U_1)(s)$. Let f be a function. If $f = (\mathrm{commute}((\mathrm{commute}(F))(s)))(x)$, then $f(i) = F'(s)(x)$.

(29) Let U_1 be a non-empty algebra over S and F be a many sorted function indexed by I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$. Let x be a set. If $x \in (\text{the sorts of } U_1)(s)$, then $(\mathrm{commute}((\mathrm{commute}(F))(s)))(x) \in \prod \mathrm{Carrier}(A,s)$.

(30) Let U_1 be a non-empty algebra over S and F be a many sorted function indexed by I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$. Then there exists a many sorted function H from U_1 into $\prod A$ such that H is a homomorphism of U_1 into $\prod A$ and for every element i of I holds $\mathrm{proj}(A,i) \circ H = F(i)$.

5. The Class of Family of Many Sorted Algebras

Let us consider I, J, S. A many sorted set indexed by I is said to be a MSAlgebra-Class of S, J if:

(Def. 4) For every set i such that $i \in I$ holds $\mathrm{it}(i)$ is an algebra family of $J(i)$ over S.

Let us consider I, S, A, E_1. The functor $\frac{A}{E_1}$ yielding a MSAlgebra-Class of S, $\mathrm{id}_{\mathrm{Classes}\, E_1}$ is defined by:

(Def. 5) For every c such that $c \in \mathrm{Classes}\, E_1$ holds $(\frac{A}{E_1})(c) = A \restriction c$.

Let us consider I, S, let J be a non-empty many sorted set indexed by I, and let C be a MSAlgebra-Class of S, J. The functor $\prod C$ yields an algebra family of I over S and is defined by the condition (Def. 6).

(Def. 6) Let given i. Suppose $i \in I$. Then there exists a non empty set J_1 and there exists an algebra family C_1 of J_1 over S such that $J_1 = J(i)$ and $C_1 = C(i)$ and $(\prod C)(i) = \prod C_1$.

The following proposition is true

(31) Let A be an algebra family of I over S and E_1 be an equivalence relation of I. Then $\prod A$ and $\prod\prod(\frac{A}{E_1})$ are isomorphic.

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