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On Polynomials with Coefficients in a Ring of Polynomials

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Summary. The main result of the paper is, that the ring of polynomials with o_1 variables and coefficients in the ring of polynomials with o_2 variables and coefficient in a ring L is isomorphic with the ring with o_1+o_2 variables, and coefficients in L.

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The articles [21], [27], [23], [13], [28], [8], [9], [20], [1], [22], [14], [24], [17], [11], [5], [10], [26], [12], [6], [2], [3], [4], [25], [7], [19], [15], [29], [18], and [16] provide the notation and terminology for this paper.

1. Preliminaries

In this paper o_1, o_2 denote ordinal numbers.

Let L_1, L_2 be non empty double loop structures. Let us note that the predicate L_1 is ring isomorphic to L_2 is reflexive. We introduce L_1 and L_2 are isomorphic as a synonym of L_1 is ring isomorphic to L_2.

Next we state the proposition

(1) Let B be a set. Suppose that for every set x holds $x \in B$ iff there exists an ordinal number o such that $x = o_1 + o$ and $o \in o_2$. Then $o_1 + o_2 = o_1 \cup B$.

Let o_1 be an ordinal number and let o_2 be a non empty ordinal number. One can check that $o_1 + o_2$ is non empty and $o_2 + o_1$ is non empty.

We now state the proposition

(2) Let n be an ordinal number and a, b be bags of n. Suppose $a < b$. Then there exists an ordinal number o such that $o \in n$ and $a(o) < b(o)$ and for every ordinal number l such that $l \in o$ holds $a(l) = b(l)$.

2. About Bags

Let o_1, o_2 be ordinal numbers, let a be an element of $\mathrm{Bags}\, o_1$, and let b be an element of $\mathrm{Bags}\, o_2$. The functor $a+b$ yields an element of $\mathrm{Bags}(o_1+o_2)$ and is defined as follows:

(Def. 1) For every ordinal number o holds if $o \in o_1$, then $(a+b)(o) = a(o)$ and if $o \in (o_1+o_2) \setminus o_1$, then $(a+b)(o) = b(o - o_1)$.

We now state several propositions:

(3) For every element a of $\mathrm{Bags}\, o_1$ and for every element b of $\mathrm{Bags}\, o_2$ such that $o_2 = \emptyset$ holds $a + b = a$.

(4) For every element a of $\mathrm{Bags}\, o_1$ and for every element b of $\mathrm{Bags}\, o_2$ such that $o_1 = \emptyset$ holds $a + b = b$.

(5) For every element b_1 of $\mathrm{Bags}\, o_1$ and for every element b_2 of $\mathrm{Bags}\, o_2$ holds $b_1 + b_2 = \mathrm{EmptyBag}(o_1 + o_2)$ iff $b_1 = \mathrm{EmptyBag}\, o_1$ and $b_2 = \mathrm{EmptyBag}\, o_2$.

(6) For every element c of $\mathrm{Bags}(o_1 + o_2)$ there exists an element c_1 of $\mathrm{Bags}\, o_1$ and there exists an element c_2 of $\mathrm{Bags}\, o_2$ such that $c = c_1 + c_2$.

(7) For all elements b_1, c_1 of $\mathrm{Bags}\, o_1$ and for all elements b_2, c_2 of $\mathrm{Bags}\, o_2$ such that $b_1 + b_2 = c_1 + c_2$ holds $b_1 = c_1$ and $b_2 = c_2$.

(8) Let n be an ordinal number, L be an Abelian add-associative right zeroed right complementable distributive associative non empty double loop structure, and p, q, r be series of n, L. Then $(p + q) * r = p * r + q * r$.

3. Main Results

Let n be an ordinal number and let L be a right zeroed Abelian add-associative right complementable unital distributive associative non trivial non empty double loop structure. Note that $\mathrm{Polynom\text{-}Ring}(n, L)$ is non trivial and distributive.

Let o_1, o_2 be non empty ordinal numbers, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let P be a polynomial of o_1, $\mathrm{Polynom\text{-}Ring}(o_2, L)$. The functor $\mathrm{Compress}\, P$ yielding a polynomial of $o_1 + o_2$, L is defined by the condition (Def. 2).

(Def. 2) Let b be an element of $\mathrm{Bags}(o_1 + o_2)$. Then there exists an element b_1 of $\mathrm{Bags}\, o_1$ and there exists an element b_2 of $\mathrm{Bags}\, o_2$ and there exists a polynomial Q_1 of o_2, L such that $Q_1 = P(b_1)$ and $b = b_1 + b_2$ and $(\mathrm{Compress}\, P)(b) = Q_1(b_2)$.

We now state several propositions:

(9) For all elements b_1, c_1 of $\mathrm{Bags}\, o_1$ and for all elements b_2, c_2 of $\mathrm{Bags}\, o_2$ such that $b_1 \mid c_1$ and $b_2 \mid c_2$ holds $b_1 + b_2 \mid c_1 + c_2$.

(10) Let b be a bag of $o_1 + o_2$, b_1 be an element of $\mathrm{Bags}\, o_1$, and b_2 be an element of $\mathrm{Bags}\, o_2$. Suppose $b \mid b_1 + b_2$. Then there exists an element c_1 of $\mathrm{Bags}\, o_1$ and there exists an element c_2 of $\mathrm{Bags}\, o_2$ such that $c_1 \mid b_1$ and $c_2 \mid b_2$ and $b = c_1 + c_2$.

(11) For all elements a_1, b_1 of $\mathrm{Bags}\, o_1$ and for all elements a_2, b_2 of $\mathrm{Bags}\, o_2$ holds $a_1 + a_2 < b_1 + b_2$ iff $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$.

(12) Let b_1 be an element of $\mathrm{Bags}\, o_1$, b_2 be an element of $\mathrm{Bags}\, o_2$, and G be a finite sequence of elements of $(\mathrm{Bags}(o_1 + o_2))^*$. Suppose that

(i) $\mathrm{dom}\, G = \mathrm{dom}\, \mathrm{divisors}\, b_1$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, \mathrm{divisors}\, b_1$ there exists an element a'_1 of $\mathrm{Bags}\, o_1$ and there exists a finite sequence F_1 of elements of $\mathrm{Bags}(o_1 + o_2)$ such that $F_1 = G_i$ and $(\mathrm{divisors}\, b_1)_i = a'_1$ and $\mathrm{len}\, F_1 = \mathrm{len}\, \mathrm{divisors}\, b_2$ and for every natural number m such that $m \in \mathrm{dom}\, F_1$ there exists an element a''_1 of $\mathrm{Bags}\, o_2$ such that $(\mathrm{divisors}\, b_2)_m = a''_1$ and $(F_1)_m = a'_1 + a''_1$.

Then $\mathrm{divisors}(b_1 + b_2) = \mathrm{Flat}(G)$.

(13) For all elements a_1, b_1, c_1 of $\mathrm{Bags}\, o_1$ and for all elements a_2, b_2, c_2 of $\mathrm{Bags}\, o_2$ such that $c_1 = b_1 -' a_1$ and $c_2 = b_2 -' a_2$ holds $(b_1 + b_2) -' (a_1 + a_2) = c_1 + c_2$.

(14) Let b_1 be an element of $\mathrm{Bags}\,o_1$, b_2 be an element of $\mathrm{Bags}\,o_2$, and G be a finite sequence of elements of $((\mathrm{Bags}(o_1+o_2))^2)^*$. Suppose that

(i) $\mathrm{dom}\,G = \mathrm{dom}\,\mathrm{decomp}\,b_1$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\,\mathrm{decomp}\,b_1$ there exist elements a'_1, b'_1 of $\mathrm{Bags}\,o_1$ and there exists a finite sequence F_1 of elements of $(\mathrm{Bags}(o_1+o_2))^2$ such that $F_1 = G_i$ and $(\mathrm{decomp}\,b_1)_i = \langle a'_1, b'_1 \rangle$ and $\mathrm{len}\,F_1 = \mathrm{len}\,\mathrm{decomp}\,b_2$ and for every natural number m such that $m \in \mathrm{dom}\,F_1$ there exist elements a''_1, b''_1 of $\mathrm{Bags}\,o_2$ such that $(\mathrm{decomp}\,b_2)_m = \langle a''_1, b''_1 \rangle$ and $(F_1)_m = \langle a'_1 + a''_1, b'_1 + b''_1 \rangle$.

Then $\mathrm{decomp}(b_1+b_2) = \mathrm{Flat}(G)$.

(15) Let o_1, o_2 be non empty ordinal numbers and L be an Abelian right zeroed add-associative right complementable unital distributive associative well unital non trivial non empty double loop structure. Then $\text{Polynom-Ring}(o_1, \text{Polynom-Ring}(o_2, L))$ and $\text{Polynom-Ring}(o_1+o_2, L)$ are isomorphic.

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