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The Evaluation of Multivariate Polynomials

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The articles [21], [10], [29], [22], [30], [32], [31], [7], [13], [3], [9], [8], [6], [11], [15], [2], [5], [19], [24], [23], [28], [4], [27], [16], [25], [26], [12], [1], [17], [18], [14], and [20] provide the notation and terminology for this paper.

1. Preliminaries

In this article we present several logical schemes. The scheme *SeqExD* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{dom} p = \operatorname{Seg}\mathcal{B}$ and for every natural number k such that $k \in \operatorname{Seg}\mathcal{B}$ holds $\mathcal{P}[k, p_k]$

provided the parameters have the following property:

- For every natural number k such that $k \in \operatorname{Seg}\mathcal{B}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[k,x]$.

The scheme *FinRecExD2* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{len} p = \mathcal{C}$ but $p_1 = \mathcal{B}$ or $\mathcal{C} = 0$ but for every natural number n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ holds $\mathcal{P}[n, p_n, p_{n+1}]$

provided the parameters have the following property:

- Let n be a natural number. Suppose $1 \leq n$ and $n \leq \mathcal{C} - 1$. Let x be an element of $\mathcal{A}$. Then there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[n,x,y]$.

The scheme *FinRecUnD2* deals with a set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a natural number $\mathcal{C}$, finite sequences $\mathcal{D}$, $\mathcal{E}$ of elements of $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> $\mathcal{D} = \mathcal{E}$

provided the following requirements are met:

- Let n be a natural number. Suppose $1 \leq n$ and $n \leq \mathcal{C} - 1$. Let x, y_1, y_2 be elements of $\mathcal{A}$. If $\mathcal{P}[n,x,y_1]$ and $\mathcal{P}[n,x,y_2]$, then $y_1 = y_2$,
- $\operatorname{len}\mathcal{D} = \mathcal{C}$ but $\mathcal{D}_1 = \mathcal{B}$ or $\mathcal{C} = 0$ but for every natural number n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ holds $\mathcal{P}[n, \mathcal{D}_n, \mathcal{D}_{n+1}]$, and
- $\operatorname{len}\mathcal{E} = \mathcal{C}$ but $\mathcal{E}_1 = \mathcal{B}$ or $\mathcal{C} = 0$ but for every natural number n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ holds $\mathcal{P}[n, \mathcal{E}_n, \mathcal{E}_{n+1}]$.

The scheme *FinInd* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

> For every natural number i such that $\mathcal{A} \leq i$ and $i \leq \mathcal{B}$ holds $\mathcal{P}[i]$

provided the following requirements are met:

- $\mathcal{P}[\mathcal{A}]$, and
- For every natural number j such that $\mathcal{A} \leq j$ and $j < \mathcal{B}$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$.

The scheme *FinInd2* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

For every natural number i such that $\mathcal{A} \leq i$ and $i \leq \mathcal{B}$ holds $\mathcal{P}[i]$

provided the following conditions are satisfied:

- $\mathcal{P}[\mathcal{A}]$, and
- Let j be a natural number. Suppose $\mathcal{A} \leq j$ and $j < \mathcal{B}$. Suppose that for every natural number j' such that $\mathcal{A} \leq j'$ and $j' \leq j$ holds $\mathcal{P}[j']$. Then $\mathcal{P}[j+1]$.

The scheme *IndFinSeq* deals with a set $\mathcal{A}$, a finite sequence $\mathcal{B}$ of elements of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every natural number i such that $1 \leq i$ and $i \leq \operatorname{len} \mathcal{B}$ holds $\mathcal{P}[\mathcal{B}(i)]$

provided the following conditions are met:

- $\mathcal{P}[\mathcal{B}(1)]$, and
- For every natural number i such that $1 \leq i$ and $i < \operatorname{len} \mathcal{B}$ holds if $\mathcal{P}[\mathcal{B}(i)]$, then $\mathcal{P}[\mathcal{B}(i+1)]$.

The following propositions are true:

(2)[1] Let L be a unital associative non empty groupoid, a be an element of L, and n, m be natural numbers. Then $\operatorname{power}_L(a, n+m) = \operatorname{power}_L(a, n) \cdot \operatorname{power}_L(a, m)$.

(3) For every well unital non empty double loop structure L holds $\mathbf{1}_L = 1_L$.

Let us observe that there exists a non empty double loop structure which is Abelian, right zeroed, add-associative, right complementable, unital, well unital, distributive, commutative, associative, and non trivial.

2. About Finite Sequences and the Functor SgmX

Next we state a number of propositions:

(4) Let p be a finite sequence and k be a natural number. Suppose $k \in \operatorname{dom} p$. Let i be a natural number. If $1 \leq i$ and $i \leq k$, then $i \in \operatorname{dom} p$.

(5) Let L be a left zeroed right zeroed non empty loop structure, p be a finite sequence of elements of the carrier of L, and i be a natural number. Suppose $i \in \operatorname{dom} p$ and for every natural number i' such that $i' \in \operatorname{dom} p$ and $i' \neq i$ holds $p_{i'} = 0_L$. Then $\sum p = p_i$.

(6) Let L be an add-associative right zeroed right complementable distributive unital non empty double loop structure and p be a finite sequence of elements of the carrier of L. If there exists a natural number i such that $i \in \operatorname{dom} p$ and $p_i = 0_L$, then $\prod p = 0_L$.

(7) Let L be an Abelian add-associative non empty loop structure, a be an element of L, and p, q be finite sequences of elements of the carrier of L. Suppose that

(i) $\operatorname{len} p = \operatorname{len} q$, and

(ii) there exists a natural number i such that $i \in \operatorname{dom} p$ and $q_i = a + p_i$ and for every natural number i' such that $i' \in \operatorname{dom} p$ and $i' \neq i$ holds $q_{i'} = p_{i'}$.

Then $\sum q = a + \sum p$.

(8) Let L be a commutative associative non empty double loop structure, a be an element of L, and p, q be finite sequences of elements of the carrier of L. Suppose that

(i) $\operatorname{len} p = \operatorname{len} q$, and

(ii) there exists a natural number i such that $i \in \operatorname{dom} p$ and $q_i = a \cdot p_i$ and for every natural number i' such that $i' \in \operatorname{dom} p$ and $i' \neq i$ holds $q_{i'} = p_{i'}$.

Then $\prod q = a \cdot \prod p$.

(9) Let X be a set, A be an empty subset of X, and R be an order in X. If R linearly orders A, then $\operatorname{SgmX}(R, A) = \emptyset$.

[1] The proposition (1) has been removed.

(10) Let X be a set, A be a finite subset of X, and R be an order in X. Suppose R linearly orders A. Let i, j be natural numbers. If $i \in \mathrm{dom}\,\mathrm{SgmX}(R,A)$ and $j \in \mathrm{dom}\,\mathrm{SgmX}(R,A)$, then if $(\mathrm{SgmX}(R,A))_i = (\mathrm{SgmX}(R,A))_j$, then $i = j$.

(11) Let X be a set, A be a finite subset of X, and a be an element of X. Suppose $a \notin A$. Let B be a finite subset of X. Suppose $B = \{a\} \cup A$. Let R be an order in X. Suppose R linearly orders B. Let k be a natural number. Suppose $k \in \mathrm{dom}\,\mathrm{SgmX}(R,B)$ and $(\mathrm{SgmX}(R,B))_k = a$. Let i be a natural number. If $1 \leq i$ and $i \leq k-1$, then $(\mathrm{SgmX}(R,B))_i = (\mathrm{SgmX}(R,A))_i$.

(12) Let X be a set, A be a finite subset of X, and a be an element of X. Suppose $a \notin A$. Let B be a finite subset of X. Suppose $B = \{a\} \cup A$. Let R be an order in X. Suppose R linearly orders B. Let k be a natural number. Suppose $k \in \mathrm{dom}\,\mathrm{SgmX}(R,B)$ and $(\mathrm{SgmX}(R,B))_k = a$. Let i be a natural number. If $k \leq i$ and $i \leq \mathrm{len}\,\mathrm{SgmX}(R,A)$, then $(\mathrm{SgmX}(R,B))_{i+1} = (\mathrm{SgmX}(R,A))_i$.

(13) Let X be a non empty set, A be a finite subset of X, and a be an element of X. Suppose $a \notin A$. Let B be a finite subset of X. Suppose $B = \{a\} \cup A$. Let R be an order in X. Suppose R linearly orders B. Let k be a natural number. If $k+1 \in \mathrm{dom}\,\mathrm{SgmX}(R,B)$ and $(\mathrm{SgmX}(R,B))_{k+1} = a$, then $\mathrm{SgmX}(R,B) = \mathrm{Ins}(\mathrm{SgmX}(R,A),k,a)$.

3. Evaluation of Bags

The following proposition is true

(14) For every set X and for every bag b of X such that $\mathrm{support}\,b = \emptyset$ holds $b = \mathrm{EmptyBag}\,X$.

Let X be a set and let b be a bag of X. We say that b is empty if and only if:

(Def. 1) $b = \mathrm{EmptyBag}\,X$.

Let X be a non empty set. Note that there exists a bag of X which is non empty.
Let X be a set and let b be a bag of X. Then $\mathrm{support}\,b$ is a finite subset of X.
We now state the proposition

(15) For every ordinal number n and for every bag b of n holds $\subseteq_n$ linearly orders $\mathrm{support}\,b$.

Let X be a set, let x be a finite sequence of elements of X, and let b be a bag of X. Then $b \cdot x$ is a partial function from $\mathbb{N}$ to $\mathbb{N}$.

Let n be an ordinal number, let b be a bag of n, let L be a non trivial unital non empty double loop structure, and let x be a function from n into L. The functor $\mathrm{eval}(b,x)$ yields an element of L and is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence y of elements of the carrier of L such that $\mathrm{len}\,y = \mathrm{len}\,\mathrm{SgmX}(\subseteq_n, \mathrm{support}\,b)$ and $\mathrm{eval}(b,x) = \prod y$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\,y$ holds $y_i = \mathrm{power}_L((x \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\,b))_i, (b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\,b))_i)$.

We now state three propositions:

(16) Let n be an ordinal number, L be a non trivial unital non empty double loop structure, and x be a function from n into L. Then $\mathrm{eval}(\mathrm{EmptyBag}\,n, x) = 1_L$.

(17) Let n be an ordinal number, L be a unital non trivial non empty double loop structure, u be a set, and b be a bag of n. If $\mathrm{support}\,b = \{u\}$, then for every function x from n into L holds $\mathrm{eval}(b,x) = \mathrm{power}_L(x(u), b(u))$.

(18) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive Abelian non trivial commutative associative non empty double loop structure, b_1, b_2 be bags of n, and x be a function from n into L. Then $\mathrm{eval}(b_1 + b_2, x) = \mathrm{eval}(b_1, x) \cdot \mathrm{eval}(b_2, x)$.

4. Evaluation of Polynomials

Let n be an ordinal number, let L be an add-associative right zeroed right complementable non empty loop structure, and let p, q be polynomials of n, L. Note that $p-q$ is finite-Support.

The following proposition is true

(19) Let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, n be an ordinal number, and p be a polynomial of n, L. If $\mathrm{Support}\, p = \emptyset$, then $p = 0_n L$.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let p be a polynomial of n, L. One can verify that $\mathrm{Support}\, p$ is finite.

Next we state the proposition

(20) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p be a polynomial of n, L. Then $\mathrm{BagOrder}\, n$ linearly orders $\mathrm{Support}\, p$.

Let n be an ordinal number and let b be an element of $\mathrm{Bags}\, n$. The functor b^{T} yields a bag of n and is defined as follows:

(Def. 3) $b^{\mathrm{T}} = b$.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let p be a polynomial of n, L, and let x be a function from n into L. The functor $\mathrm{eval}(p,x)$ yielding an element of L is defined by the condition (Def. 4).

(Def. 4) There exists a finite sequence y of elements of the carrier of L such that $\mathrm{len}\, y = \mathrm{len}\, \mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p)$ and $\mathrm{eval}(p,x) = \sum y$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, y$ holds $y_i = (p \cdot \mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p))_i \cdot \mathrm{eval}(((\mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p))_i)^{\mathrm{T}}, x)$.

The following propositions are true:

(21) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, p be a polynomial of n, L, and b be a bag of n. If $\mathrm{Support}\, p = \{b\}$, then for every function x from n into L holds $\mathrm{eval}(p,x) = p(b) \cdot \mathrm{eval}(b,x)$.

(22) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and x be a function from n into L. Then $\mathrm{eval}(0_n L, x) = 0_L$.

(23) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and x be a function from n into L. Then $\mathrm{eval}(1\_(n,L),x) = 1_L$.

(24) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, p be a polynomial of n, L, and x be a function from n into L. Then $\mathrm{eval}(-p,x) = -\mathrm{eval}(p,x)$.

(25) Let n be an ordinal number, L be a right zeroed add-associative right complementable Abelian unital distributive non trivial non empty double loop structure, p, q be polynomials of n, L, and x be a function from n into L. Then $\mathrm{eval}(p+q,x) = \mathrm{eval}(p,x) + \mathrm{eval}(q,x)$.

(26) Let n be an ordinal number, L be a right zeroed add-associative right complementable Abelian unital distributive non trivial non empty double loop structure, p, q be polynomials of n, L, and x be a function from n into L. Then $\mathrm{eval}(p-q,x) = \mathrm{eval}(p,x) - \mathrm{eval}(q,x)$.

(27) Let n be an ordinal number, L be a right zeroed add-associative right complementable Abelian unital distributive non trivial commutative associative non empty double loop structure, p, q be polynomials of n, L, and x be a function from n into L. Then $\mathrm{eval}(p * q, x) = \mathrm{eval}(p, x) \cdot \mathrm{eval}(q, x)$.

5. Evaluation Homomorphism

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let x be a function from n into L. The functor Polynom-Evaluation(n, L, x) yielding a map from Polynom-Ring(n, L) into L is defined as follows:

(Def. 5) For every polynomial p of n, L holds $(\text{Polynom-Evaluation}(n, L, x))(p) = \mathrm{eval}(p, x)$.

Let n be an ordinal number and let L be a right zeroed Abelian add-associative right complementable well unital distributive associative non trivial non empty double loop structure. One can check that Polynom-Ring(n, L) is well unital.

Let n be an ordinal number, let L be an Abelian right zeroed add-associative right complementable well unital distributive associative non trivial non empty double loop structure, and let x be a function from n into L. Note that Polynom-Evaluation(n, L, x) is unity-preserving.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable Abelian unital distributive non trivial non empty double loop structure, and let x be a function from n into L. Note that Polynom-Evaluation(n, L, x) is additive.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable Abelian unital distributive non trivial commutative associative non empty double loop structure, and let x be a function from n into L. One can check that Polynom-Evaluation(n, L, x) is multiplicative.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable Abelian well unital distributive non trivial commutative associative non empty double loop structure, and let x be a function from n into L. Observe that Polynom-Evaluation(n, L, x) is ring homomorphism.

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