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Solving Roots of Polynomial Equation of Degree 4 with Real Coefficients

Xiquan Liang
Qingdao University of Science and Technology

Summary. In this paper, we describe the definition of the fourth degree algebraic equations and their properties. We clarify the relation between the four roots of this equation and its coefficient. Also, the form of these roots for various conditions is discussed. This solution is known as the Cardano solution.

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The articles [1], [4], [2], and [3] provide the notation and terminology for this paper.

Let a, b, c, d, e, x be real numbers. The functor $\mathrm{Four}(a,b,c,d,e,x)$ is defined as follows:

(Def. 1) $\mathrm{Four}(a,b,c,d,e,x) = a \cdot x^4 + b \cdot x^3 + c \cdot x^{\mathbf{2}} + d \cdot x + e$.

Let a, b, c, d, e, x be real numbers. Observe that $\mathrm{Four}(a,b,c,d,e,x)$ is real.

We now state several propositions:

(1) Let a, c, e, x be real numbers. Suppose $a \neq 0$ and $e \neq 0$ and $c^{\mathbf{2}} - 4 \cdot a \cdot e > 0$. Suppose $\mathrm{Four}(a,0,c,0,e,x) = 0$. Then $x \neq 0$ but $x = \sqrt{\frac{-c+\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = \sqrt{\frac{-c-\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = -\sqrt{\frac{-c+\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = -\sqrt{\frac{-c-\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$.

(2) Let a, b, c, x, y be real numbers. Suppose $a \neq 0$ and $y = x + \frac{1}{x}$. If $\mathrm{Four}(a,b,c,b,a,x) = 0$, then $x \neq 0$ and $(a \cdot y^{\mathbf{2}} + b \cdot y + c) - 2 \cdot a = 0$.

(3) Let a, b, c, x, y be real numbers. Suppose $a \neq 0$ and $(b^{\mathbf{2}} - 4 \cdot a \cdot c) + 8 \cdot a^{\mathbf{2}} > 0$ and $y = x + \frac{1}{x}$. Suppose $\mathrm{Four}(a,b,c,b,a,x) = 0$. Let y_1, y_2 be real numbers. Suppose $y_1 = \frac{-b+\sqrt{(b^{\mathbf{2}}-4 \cdot a \cdot c)+8 \cdot a^{\mathbf{2}}}}{2 \cdot a}$ and $y_2 = \frac{-b-\sqrt{(b^{\mathbf{2}}-4 \cdot a \cdot c)+8 \cdot a^{\mathbf{2}}}}{2 \cdot a}$. Then $x \neq 0$ but $x = \frac{y_1+\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2+\sqrt{\Delta(1,-y_2,1)}}{2}$ or $x = \frac{y_1-\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2-\sqrt{\Delta(1,-y_2,1)}}{2}$.

(4) For every real number x holds $x^3 = x^{\mathbf{2}} \cdot x$ and $x^3 \cdot x = x^4$ and $x^{\mathbf{2}} \cdot x^{\mathbf{2}} = x^4$.

(5) For all real numbers x, y such that $x + y \neq 0$ holds $(x+y)^4 = (x^3 + (3 \cdot y \cdot x^{\mathbf{2}} + 3 \cdot y^{\mathbf{2}} \cdot x) + y^3) \cdot x + (x^3 + (3 \cdot y \cdot x^{\mathbf{2}} + 3 \cdot y^{\mathbf{2}} \cdot x) + y^3) \cdot y$.

(6) For all real numbers x, y such that $x + y \neq 0$ holds $(x+y)^4 = x^4 + (4 \cdot y \cdot x^3 + 6 \cdot y^{\mathbf{2}} \cdot x^{\mathbf{2}} + 4 \cdot y^3 \cdot x) + y^4$.

(7) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1,a_2,a_3,a_4,a_5,x) = \mathrm{Four}(b_1,b_2,b_3,b_4,b_5,x)$. Then $a_5 = b_5$ and $((a_1 - a_2) + a_3) - a_4 = ((b_1 - b_2) + b_3) - b_4$ and $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$.

(8) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1,a_2,a_3,a_4,a_5,x) = \mathrm{Four}(b_1,b_2,b_3,b_4,b_5,x)$. Then $a_1 - b_1 = b_3 - a_3$ and $a_2 - b_2 = b_4 - a_4$.

(9) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1,a_2,a_3,a_4,a_5,x) = \mathrm{Four}(b_1,b_2,b_3,b_4,b_5,x)$. Then $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$ and $a_4 = b_4$ and $a_5 = b_5$.

Let a_1, x_1, x_2, x_3, x_4, x be real numbers. The functor $\mathrm{Four0}(a_1,x_1,x_2,x_3,x_4,x)$ is defined as follows:

(Def. 2) $\mathrm{Four0}(a_1,x_1,x_2,x_3,x_4,x) = a_1 \cdot ((x-x_1) \cdot (x-x_2) \cdot (x-x_3) \cdot (x-x_4))$.

Let a_1, x_1, x_2, x_3, x_4, x be real numbers. One can check that $\mathrm{Four0}(a_1,x_1,x_2,x_3,x_4,x)$ is real.

Next we state four propositions:

(10) Let a_1, a_2, a_3, a_4, a_5, x, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$. Suppose that for every real number x holds $\mathrm{Four}(a_1,a_2,a_3,a_4,a_5,x) = \mathrm{Four0}(a_1,x_1,x_2,x_3,x_4,x)$. Then $\frac{a_1 \cdot x^4 + a_2 \cdot x^3 + a_3 \cdot x^{\mathbf{2}} + a_4 \cdot x + a_5}{a_1} = ((x^{\mathbf{2}} \cdot x^{\mathbf{2}} - (x_1+x_2+x_3) \cdot (x^{\mathbf{2}} \cdot x)) + (x_1 \cdot x_3 + x_2 \cdot x_3 + x_1 \cdot x_2) \cdot x^{\mathbf{2}}) - x_1 \cdot x_2 \cdot x_3 \cdot x - (x-x_1) \cdot (x-x_2) \cdot (x-x_3) \cdot x_4$.

(11) Let a_1, a_2, a_3, a_4, a_5, x, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$. Suppose that for every real number x holds $\mathrm{Four}(a_1,a_2,a_3,a_4,a_5,x) = \mathrm{Four0}(a_1,x_1,x_2,x_3,x_4,x)$. Then $\frac{a_1 \cdot x^4 + a_2 \cdot x^3 + a_3 \cdot x^{\mathbf{2}} + a_4 \cdot x + a_5}{a_1} = (((x^4 - (x_1+x_2+x_3+x_4) \cdot x^3) + (x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + (x_2 \cdot x_3 + x_2 \cdot x_4) + x_3 \cdot x_4) \cdot x^{\mathbf{2}}) - (x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4) \cdot x) + x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

(12) Let a_1, a_2, a_3, a_4, a_5, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$ and for every real number x holds $\mathrm{Four}(a_1,a_2,a_3,a_4,a_5,x) = \mathrm{Four0}(a_1,x_1,x_2,x_3,x_4,x)$. Then $\frac{a_2}{a_1} = -(x_1+x_2+x_3+x_4)$ and $\frac{a_3}{a_1} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + (x_2 \cdot x_3 + x_2 \cdot x_4) + x_3 \cdot x_4$ and $\frac{a_4}{a_1} = -(x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4)$ and $\frac{a_5}{a_1} = x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

(13) Let a, k, y be real numbers. Suppose $a \neq 0$. Suppose that for every real number x holds $x^4 + a^4 = k \cdot a \cdot x \cdot (x^{\mathbf{2}} + a^{\mathbf{2}})$. Then $(y^4 - k \cdot y^3 - k \cdot y) + 1 = 0$.

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