JOURNAL OF FORMALIZED MATHEMATICS
Volume 12, Released 2000, Published 2003
Inst. of Computer Science, Univ. of Białystok

Solving Roots of Polynomial Equations of Degree 2 and 3 with Real Coefficients

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Summary. In this paper, we describe the definition of the first, second, and third degree algebraic equations and their properties. In Section 1, we defined the simple first-degree and second-degree (quadratic) equation and discussed the relation between the roots of each equation and their coefficients. Also, we clarified the form of the root within the range of real numbers. Furthermore, the extraction of the root using the discriminant of equation is clarified. In Section 2, we defined the third-degree (cubic) equation and clarified the relation between the three roots of this equation and its coefficient. Also, the form of these roots for various conditions is discussed. This solution is known as the Cardano solution.

MML Identifier: `POLYEQ_1`.

WWW: `http://mizar.org/JFM/Vol12/polyeq_1.html`

The articles [1], [4], [3], and [2] provide the notation and terminology for this paper.

1. Equation of Degree 1 and 2

In this paper a, a', a_1, a_2, a_3, b, b', c, c', d, d', h, p, q, x, x_1, x_2, x_3, u, v, y are real numbers.

Let a, b, x be real numbers. The functor $\mathrm{Poly1}(a,b,x)$ is defined as follows:

(Def. 1) $\mathrm{Poly1}(a,b,x) = a \cdot x + b$.

Let a, b, x be real numbers. Observe that $\mathrm{Poly1}(a,b,x)$ is real.

Let a, b, x be real numbers. Then $\mathrm{Poly1}(a,b,x)$ is a real number.

The following three propositions are true:

(1) If $a \neq 0$ and $\mathrm{Poly1}(a,b,x) = 0$, then $x = -\frac{b}{a}$.

(2) $\mathrm{Poly1}(0,0,x) = 0$.

(3) If $b \neq 0$, then it is not true that there exists x such that $\mathrm{Poly1}(0,b,x) = 0$.

Let a, b, c, x be real numbers. The functor $\mathrm{Poly2}(a,b,c,x)$ is defined as follows:

(Def. 2) $\mathrm{Poly2}(a,b,c,x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$.

Let a, b, c, x be real numbers. Note that $\mathrm{Poly2}(a,b,c,x)$ is real.

Let a, b, c, x be real numbers. Then $\mathrm{Poly2}(a,b,c,x)$ is a real number.

Next we state several propositions:

(4) If for every x holds $\mathrm{Poly2}(a,b,c,x) = \mathrm{Poly2}(a',b',c',x)$, then $a = a'$ and $b = b'$ and $c = c'$.

(5) If $a \neq 0$ and $\Delta(a,b,c) \geq 0$, then for every x such that $\mathrm{Poly2}(a,b,c,x) = 0$ holds $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(6) If $a \neq 0$ and $\Delta(a,b,c) = 0$ and $\mathrm{Poly2}(a,b,c,x) = 0$, then $x = -\frac{b}{2 \cdot a}$.

(7) If $a \neq 0$ and $\Delta(a,b,c) < 0$, then it is not true that there exists x such that $\mathrm{Poly2}(a,b,c,x) = 0$.

(8) If $b \neq 0$ and for every x holds $\mathrm{Poly2}(0,b,c,x) = 0$, then $x = -\frac{c}{b}$.

(9) $\mathrm{Poly2}(0,0,0,x) = 0$.

(10) If $c \neq 0$, then it is not true that there exists x such that $\mathrm{Poly2}(0,0,c,x) = 0$.

Let a, x, x_1, x_2 be real numbers. The functor $\mathrm{Quard}(a,x_1,x_2,x)$ is defined by:

(Def. 3) $\mathrm{Quard}(a,x_1,x_2,x) = a \cdot ((x - x_1) \cdot (x - x_2))$.

Let a, x, x_1, x_2 be real numbers. Observe that $\mathrm{Quard}(a,x_1,x_2,x)$ is real.
Let a, x, x_1, x_2 be real numbers. Then $\mathrm{Quard}(a,x_1,x_2,x)$ is a real number.
Next we state the proposition

(11) If $a \neq 0$ and for every x holds $\mathrm{Poly2}(a,b,c,x) = \mathrm{Quard}(a,x_1,x_2,x)$, then $\frac{b}{a} = -(x_1 + x_2)$ and $\frac{c}{a} = x_1 \cdot x_2$.

2. EQUATION OF DEGREE 3

Let a, b, c, d, x be real numbers. The functor $\mathrm{Poly3}(a,b,c,d,x)$ is defined as follows:

(Def. 4) $\mathrm{Poly3}(a,b,c,d,x) = a \cdot x^3 + b \cdot x^{\mathbf{2}} + c \cdot x + d$.

Let a, b, c, d, x be real numbers. Observe that $\mathrm{Poly3}(a,b,c,d,x)$ is real.
Let a, b, c, d, x be real numbers. Then $\mathrm{Poly3}(a,b,c,d,x)$ is a real number.
The following proposition is true

(12) If for every x holds $\mathrm{Poly3}(a,b,c,d,x) = \mathrm{Poly3}(a',b',c',d',x)$, then $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$.

Let a, x, x_1, x_2, x_3 be real numbers. The functor $\mathrm{Tri}(a,x_1,x_2,x_3,x)$ is defined by:

(Def. 5) $\mathrm{Tri}(a,x_1,x_2,x_3,x) = a \cdot ((x - x_1) \cdot (x - x_2) \cdot (x - x_3))$.

Let a, x, x_1, x_2, x_3 be real numbers. Observe that $\mathrm{Tri}(a,x_1,x_2,x_3,x)$ is real.
Let a, x, x_1, x_2, x_3 be real numbers. Then $\mathrm{Tri}(a,x_1,x_2,x_3,x)$ is a real number.
Next we state a number of propositions:

(13) If $a \neq 0$ and for every x holds $\mathrm{Poly3}(a,b,c,d,x) = \mathrm{Tri}(a,x_1,x_2,x_3,x)$, then $\frac{b}{a} = -(x_1 + x_2 + x_3)$ and $\frac{c}{a} = x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3$ and $\frac{d}{a} = -x_1 \cdot x_2 \cdot x_3$.

(14) $(y + h)^3 = y^3 + (3 \cdot h \cdot y^{\mathbf{2}} + 3 \cdot h^{\mathbf{2}} \cdot y) + h^3$.

(15) Suppose $a \neq 0$ and $\mathrm{Poly3}(a,b,c,d,x) = 0$. Let given a_1, a_2, a_3, h, y. Suppose $y = x + \frac{b}{3 \cdot a}$ and $h = -\frac{b}{3 \cdot a}$ and $a_1 = \frac{b}{a}$ and $a_2 = \frac{c}{a}$ and $a_3 = \frac{d}{a}$. Then $y^3 + ((3 \cdot h + a_1) \cdot y^{\mathbf{2}} + (3 \cdot h^{\mathbf{2}} + 2 \cdot (a_1 \cdot h) + a_2) \cdot y) + (h^3 + a_1 \cdot h^{\mathbf{2}} + (a_2 \cdot h + a_3)) = 0$.

(16) Suppose $a \neq 0$ and $\mathrm{Poly3}(a,b,c,d,x) = 0$. Let given a_1, a_2, a_3, h, y. Suppose $y = x + \frac{b}{3 \cdot a}$ and $h = -\frac{b}{3 \cdot a}$ and $a_1 = \frac{b}{a}$ and $a_2 = \frac{c}{a}$ and $a_3 = \frac{d}{a}$. Then $y^3 + 0 \cdot y^{\mathbf{2}} + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} \cdot y + (2 \cdot (\frac{b}{3 \cdot a})^3 + \frac{3 \cdot a \cdot d - b \cdot c}{3 \cdot a^{\mathbf{2}}}) = 0$.

(17) Suppose $y^3 + 0 \cdot y^{\mathbf{2}} + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} \cdot y + (2 \cdot (\frac{b}{3 \cdot a})^3 + \frac{3 \cdot a \cdot d - b \cdot c}{3 \cdot a^{\mathbf{2}}}) = 0$. Let given p, q. If $p = \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}}$ and $q = 2 \cdot (\frac{b}{3 \cdot a})^3 + \frac{3 \cdot a \cdot d - b \cdot c}{3 \cdot a^{\mathbf{2}}}$, then $\mathrm{Poly3}(1,0,p,q,y) = 0$.

(18) If $\mathrm{Poly3}(1,0,p,q,y)=0$, then for all u, v such that $y=u+v$ and $3\cdot v\cdot u+p=0$ holds $u^3+v^3=-q$ and $u^3\cdot v^3=(-\frac{p}{3})^3$.

(19) Suppose $\mathrm{Poly3}(1,0,p,q,y)=0$. Let given u, v. Suppose $y=u+v$ and $3\cdot v\cdot u+p=0$. Then

(i) $y=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+(\frac{p}{3})^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+(\frac{p}{3})^3}}$, or

(ii) $y=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+(\frac{p}{3})^3}}+\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+(\frac{p}{3})^3}}$, or

(iii) $y=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+(\frac{p}{3})^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+(\frac{p}{3})^3}}$.

(20) If $b\neq 0$ and $\Delta(b,c,d)>0$ and $\mathrm{Poly3}(0,b,c,d,x)=0$, then $x=\frac{-c+\sqrt{\Delta(b,c,d)}}{2\cdot b}$ or $x=\frac{-c-\sqrt{\Delta(b,c,d)}}{2\cdot b}$.

(21) Suppose $a\neq 0$ and $p=\frac{c}{a}$ and $q=\frac{d}{a}$ and $\mathrm{Poly3}(a,0,c,d,x)=0$. Let given u, v. Suppose $x=u+v$ and $3\cdot v\cdot u+p=0$. Then

(i) $x=\sqrt[3]{-\frac{d}{2\cdot a}+\sqrt{\frac{d^2}{4\cdot a^2}+(\frac{c}{3\cdot a})^3}}+\sqrt[3]{-\frac{d}{2\cdot a}-\sqrt{\frac{d^2}{4\cdot a^2}+(\frac{c}{3\cdot a})^3}}$, or

(ii) $x=\sqrt[3]{-\frac{d}{2\cdot a}+\sqrt{\frac{d^2}{4\cdot a^2}+(\frac{c}{3\cdot a})^3}}+\sqrt[3]{-\frac{d}{2\cdot a}+\sqrt{\frac{d^2}{4\cdot a^2}+(\frac{c}{3\cdot a})^3}}$, or

(iii) $x=\sqrt[3]{-\frac{d}{2\cdot a}-\sqrt{\frac{d^2}{4\cdot a^2}+(\frac{c}{3\cdot a})^3}}+\sqrt[3]{-\frac{d}{2\cdot a}-\sqrt{\frac{d^2}{4\cdot a^2}+(\frac{c}{3\cdot a})^3}}$.

(22) If $a\neq 0$ and $\Delta(a,b,c)\geq 0$ and $\mathrm{Poly3}(a,b,c,0,x)=0$, then $x=0$ or $x=\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $x=\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}$.

(23) If $a\neq 0$ and $\frac{c}{a}<0$ and $\mathrm{Poly3}(a,0,c,0,x)=0$, then $x=0$ or $x=\sqrt{-\frac{c}{a}}$ or $x=-\sqrt{-\frac{c}{a}}$.

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Received May 18, 2000

Published January 2, 2004