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Processes in Petri nets

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Summary. Sequential and concurrent compositions of processes in Petri nets are introduced. A process is formalized as a set of (possible), so called, firing sequences. In the definition of the sequential composition the standard concatenation is used

$$R_1 \operatorname{before} R_2 = \{p_1 \frown p_2 : p_1 \in R_1 \ \wedge\ p_2 \in R_2\}$$

The definition of the concurrent composition is more complicated

$$R_1 \operatorname{concur} R_2 = \{q_1 \cup q_2 : q_1 \text{ misses } q_2 \ \wedge\ \operatorname{Seq} q_1 \in R_1 \ \wedge\ \operatorname{Seq} q_2 \in R_2\}$$

For example,

$$\{\langle t_0 \rangle\} \operatorname{concur} \{\langle t_1, t_2 \rangle\} = \{\langle t_0, t_1, t_2 \rangle, \langle t_1, t_0, t_2 \rangle, \langle t_1, t_2, t_0 \rangle\}$$

The basic properties of the compositions are shown.

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The articles [13], [12], [18], [5], [17], [9], [1], [3], [6], [11], [15], [2], [14], [7], [16], [8], [10], and [4] provide the notation and terminology for this paper.

1. PRELIMINARIES

We use the following convention: i denotes a natural number and x, x_1, x_2, y_1, y_2 denote sets.

The following propositions are true:

(1) If $i > 0$, then $\{\langle i, x \rangle\}$ is a finite subsequence.

(2) For every finite subsequence q holds $q = \emptyset$ iff $\operatorname{Seq} q = \emptyset$.

(3) For every finite subsequence q such that $q = \{\langle i, x \rangle\}$ holds $\operatorname{Seq} q = \langle x \rangle$.

Let us observe that every finite subsequence is finite.

We now state several propositions:

(4) For every finite subsequence q such that $\operatorname{Seq} q = \langle x \rangle$ there exists i such that $q = \{\langle i, x \rangle\}$.

(5) If $\{\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle\}$ is a finite sequence, then $x_1 = 1$ and $x_2 = 1$ and $y_1 = y_2$ or $x_1 = 1$ and $x_2 = 2$ or $x_1 = 2$ and $x_2 = 1$.

(6) $\langle x_1, x_2\rangle = \{\langle 1, x_1\rangle, \langle 2, x_2\rangle\}$.

(7) For every finite subsequence p holds $\overline{\overline{p}} = \operatorname{len}\operatorname{Seq} p$.

(8) For all binary relations P, R such that $\operatorname{dom} P$ misses $\operatorname{dom} R$ holds P misses R.

(9) For all sets X, Y and for all binary relations P, R such that X misses Y holds $P{\restriction}X$ misses $R{\restriction}Y$.

(10) For all functions f, g, h such that $f \subseteq h$ and $g \subseteq h$ and f misses g holds $\operatorname{dom} f$ misses $\operatorname{dom} g$.

(11) For every set Y and for every binary relation R holds $Y{\restriction}R \subseteq R{\restriction}R^{-1}(Y)$.

(12) For every set Y and for every function f holds $Y{\restriction}f = f{\restriction}f^{-1}(Y)$.

2. Markings on Petri Nets

Let P be a set. A function is called a marking of P if:

(Def. 1) $\operatorname{dom} \mathrm{it} = P$ and $\operatorname{rng} \mathrm{it} \subseteq \mathbb{N}$.

We adopt the following rules: P, p, x denote sets, m_1, m_2, m_3, m_4, m denote markings of P, and i, j, j_1, k denote natural numbers.

Let P be a set, let m be a marking of P, and let p be a set. Then $m(p)$ is a natural number. We introduce the m multitude of p as a synonym of $m(p)$.

The scheme *MarkingLambda* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a natural number, and states that:

There exists a marking m of $\mathcal{A}$ such that for every p such that $p \in \mathcal{A}$ holds the m multitude of $p = \mathcal{F}(p)$

for all values of the parameters.

Let us consider P, m_1, m_2. Let us observe that $m_1 = m_2$ if and only if:

(Def. 2) For every p such that $p \in P$ holds the m_1 multitude of $p =$ the m_2 multitude of p.

Let us consider P. The functor $\{\}_P$ yields a marking of P and is defined as follows:

(Def. 3) $\{\}_P = P \longmapsto 0$.

Let P be a set and let m_1, m_2 be markings of P. The predicate $m_1 \subseteq m_2$ is defined as follows:

(Def. 4) For every set p such that $p \in P$ holds the m_1 multitude of $p \leq$ the m_2 multitude of p.

Let us note that the predicate $m_1 \subseteq m_2$ is reflexive.

The following propositions are true:

(13) $\{\}_P \subseteq m$.

(14) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_3$, then $m_1 \subseteq m_3$.

Let P be a set and let m_1, m_2 be markings of P. The functor $m_1 + m_2$ yields a marking of P and is defined as follows:

(Def. 5) For every set p such that $p \in P$ holds the $m_1 + m_2$ multitude of $p =$ (the m_1 multitude of p) $+$ (the m_2 multitude of p).

Let us note that the functor $m_1 + m_2$ is commutative.

Next we state the proposition

(15) $m + \{\}_P = m$.

Let P be a set and let m_1, m_2 be markings of P. Let us assume that $m_2 \subseteq m_1$. The functor $m_1 - m_2$ yields a marking of P and is defined as follows:

(Def. 6) For every set p such that $p \in P$ holds the $m_1 - m_2$ multitude of $p =$ (the m_1 multitude of $p) -$ (the m_2 multitude of p).

We now state a number of propositions:

(16) $m_1 \subseteq m_1 + m_2$.

(17) $m - \{\}_P = m$.

(18) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_3$, then $m_3 - m_2 \subseteq m_3 - m_1$.

(19) $(m_1 + m_2) - m_2 = m_1$.

(20) If $m \subseteq m_1$ and $m_1 \subseteq m_2$, then $m_1 - m \subseteq m_2 - m$.

(21) If $m_1 \subseteq m_2$, then $(m_2 + m_3) - m_1 = (m_2 - m_1) + m_3$.

(22) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_1$, then $m_1 = m_2$.

(23) $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$.

(24) If $m_1 \subseteq m_2$ and $m_3 \subseteq m_4$, then $m_1 + m_3 \subseteq m_2 + m_4$.

(25) If $m_1 \subseteq m_2$, then $m_2 - m_1 \subseteq m_2$.

(26) If $m_1 \subseteq m_2$ and $m_3 \subseteq m_4$ and $m_4 \subseteq m_1$, then $m_1 - m_4 \subseteq m_2 - m_3$.

(27) If $m_1 \subseteq m_2$, then $m_2 = (m_2 - m_1) + m_1$.

(28) $(m_1 + m_2) - m_1 = m_2$.

(29) If $m_2 + m_3 \subseteq m_1$, then $m_1 - m_2 - m_3 = m_1 - (m_2 + m_3)$.

(30) If $m_3 \subseteq m_2$ and $m_2 \subseteq m_1$, then $m_1 - (m_2 - m_3) = (m_1 - m_2) + m_3$.

(31) $m \in \mathbb{N}^P$.

(32) If $x \in \mathbb{N}^P$, then x is a marking of P.

3. Transitions and Firing

Let us consider P. Transition of P is defined by:

(Def. 7) There exist m_1, m_2 such that it $= \langle m_1, m_2 \rangle$.

In the sequel t, t_1, t_2 denote transitions of P.

Let us consider P, t. Then $t_{\mathbf{1}}$ is a marking of P. We introduce Pre t as a synonym of $t_{\mathbf{1}}$. Then $t_{\mathbf{2}}$ is a marking of P. We introduce Post t as a synonym of $t_{\mathbf{2}}$.

Let us consider P, m, t. The functor $\mathrm{fire}(t, m)$ yields a marking of P and is defined by:

(Def. 8) $\mathrm{fire}(t,m) = \begin{cases} (m - \mathrm{Pre}\,t) + \mathrm{Post}\,t, \text{ if } \mathrm{Pre}\,t \subseteq m, \\ m, \text{ otherwise.} \end{cases}$

The following proposition is true

(33) If $\mathrm{Pre}\,t_1 + \mathrm{Pre}\,t_2 \subseteq m$, then $\mathrm{fire}(t_2, \mathrm{fire}(t_1, m)) = (m - \mathrm{Pre}\,t_1 - \mathrm{Pre}\,t_2) + \mathrm{Post}\,t_1 + \mathrm{Post}\,t_2$.

Let us consider P, t. The functor fire t yielding a function is defined by:

(Def. 9) $\mathrm{dom}\,\mathrm{fire}\,t = \mathbb{N}^P$ and for every marking m of P holds $(\mathrm{fire}\,t)(m) = \mathrm{fire}(t, m)$.

We now state two propositions:

(34) $\mathrm{rng}\,\mathrm{fire}\,t \subseteq \mathbb{N}^P$.

(35) $\text{fire}(t_2, \text{fire}(t_1, m)) = (\text{fire}\, t_2 \cdot \text{fire}\, t_1)(m)$.

Let us consider P. A non empty set is called a Petri net over P if:

(Def. 10) For every set x such that $x \in$ it holds x is a transition of P.

In the sequel N is a Petri net over P.
Let us consider P, N. We see that the element of N is a transition of P.
In the sequel e, e_1, e_2 are elements of N.

4. Firing Sequences of Transitions

Let us consider P, N. A firing-sequence of N is an element of N^*.
In the sequel C, C_1, C_2 denote firing-sequences of N.
Let us consider P, N, C. The functor fire C yields a function and is defined by the condition (Def. 11).

(Def. 11) There exists a function yielding finite sequence F such that $\text{fire}\, C = \text{compose}_{\mathbb{N}^P} F$ and $\text{len}\, F = \text{len}\, C$ and for every natural number i such that $i \in \text{dom}\, C$ holds $F(i) = \text{fire}\, (C_i$ **qua** element of $N)$.

The following propositions are true:

(36) $\text{fire}\, (\varepsilon_N) = \text{id}_{\mathbb{N}^P}$.

(37) $\text{fire}\, \langle e \rangle = \text{fire}\, e$.

(38) $\text{fire}\, e \cdot \text{id}_{\mathbb{N}^P} = \text{fire}\, e$.

(39) $\text{fire}\, \langle e_1, e_2 \rangle = \text{fire}\, e_2 \cdot \text{fire}\, e_1$.

(40) $\text{dom}\, \text{fire}\, C = \mathbb{N}^P$ and $\text{rng}\, \text{fire}\, C \subseteq \mathbb{N}^P$.

(41) $\text{fire}\, (C_1 \frown C_2) = \text{fire}\, C_2 \cdot \text{fire}\, C_1$.

(42) $\text{fire}\, (C \frown \langle e \rangle) = \text{fire}\, e \cdot \text{fire}\, C$.

Let us consider P, N, C, m. The functor $\text{fire}(C, m)$ yielding a marking of P is defined by:

(Def. 12) $\text{fire}(C, m) = (\text{fire}\, C)(m)$.

5. Sequential Composition

Let us consider P, N. A process in N is a subset of N^*.
In the sequel R, R_1, R_2, R_3, P_1, P_2 denote processes in N.
One can check that every function which is finite sequence-like is also finite subsequence-like.
Let us consider P, N, R_1, R_2. The functor R_1 before R_2 yielding a process in N is defined by:

(Def. 13) $R_1\, \text{before}\, R_2 = \{C_1 \frown C_2 : C_1 \in R_1 \land C_2 \in R_2\}$.

Let us consider P, N and let R_1, R_2 be non empty processes in N. Note that R_1 before R_2 is non empty.
We now state several propositions:

(43) $(R_1 \cup R_2)\, \text{before}\, R = (R_1\, \text{before}\, R) \cup (R_2\, \text{before}\, R)$.

(44) $R\, \text{before}\, (R_1 \cup R_2) = (R\, \text{before}\, R_1) \cup (R\, \text{before}\, R_2)$.

(45) $\{C_1\}\, \text{before}\, \{C_2\} = \{C_1 \frown C_2\}$.

(46) $\{C_1, C_2\}\, \text{before}\, \{C\} = \{C_1 \frown C, C_2 \frown C\}$.

(47) $\{C\}\, \text{before}\, \{C_1, C_2\} = \{C \frown C_1, C \frown C_2\}$.

6. Concurrent Composition

Let us consider P, N, R_1, R_2. The functor $R_1 \operatorname{concur} R_2$ yielding a process in N is defined as follows:

(Def. 14) $R_1 \operatorname{concur} R_2 = \{C : \bigvee_{q_1, q_2 : \text{finite subsequence}} (C = q_1 \cup q_2 \wedge q_1 \text{ misses } q_2 \wedge \operatorname{Seq} q_1 \in R_1 \wedge \operatorname{Seq} q_2 \in R_2)\}$.

Let us note that the functor $R_1 \operatorname{concur} R_2$ is commutative.

One can prove the following four propositions:

(48) $(R_1 \cup R_2) \operatorname{concur} R = (R_1 \operatorname{concur} R) \cup (R_2 \operatorname{concur} R)$.

(49) $\{\langle e_1 \rangle\} \operatorname{concur} \{\langle e_2 \rangle\} = \{\langle e_1, e_2 \rangle, \langle e_2, e_1 \rangle\}$.

(50) $\{\langle e_1 \rangle, \langle e_2 \rangle\} \operatorname{concur} \{\langle e \rangle\} = \{\langle e_1, e \rangle, \langle e_2, e \rangle, \langle e, e_1 \rangle, \langle e, e_2 \rangle\}$.

(51) $(R_1 \operatorname{before} R_2) \operatorname{before} R_3 = R_1 \operatorname{before} (R_2 \operatorname{before} R_3)$.

Let p be a finite subsequence and let i be a natural number. The functor $\operatorname{Shift}^i p$ yielding a finite subsequence is defined as follows:

(Def. 15) $\operatorname{dom} \operatorname{Shift}^i p = \{i + k; k$ ranges over natural numbers: $k \in \operatorname{dom} p\}$ and for every natural number j such that $j \in \operatorname{dom} p$ holds $(\operatorname{Shift}^i p)(i + j) = p(j)$.

In the sequel q, q_1, q_2 are finite subsequences.

Next we state a number of propositions:

(52) $\operatorname{Shift}^0 q = q$.

(53) $\operatorname{Shift}^{i+j} q = \operatorname{Shift}^i \operatorname{Shift}^j q$.

(54) For every finite sequence p such that $p \neq \emptyset$ holds $\operatorname{dom} \operatorname{Shift}^i p = \{j_1 : i + 1 \leq j_1 \wedge j_1 \leq i + \operatorname{len} p\}$.

(55) For every finite subsequence q holds $q = \emptyset$ iff $\operatorname{Shift}^i q = \emptyset$.

(56) Let q be a finite subsequence. Then there exists a finite subsequence s_1 such that $\operatorname{dom} s_1 = \operatorname{dom} q$ and $\operatorname{rng} s_1 = \operatorname{dom} \operatorname{Shift}^i q$ and for every k such that $k \in \operatorname{dom} q$ holds $s_1(k) = i + k$ and s_1 is one-to-one.

(57) For every finite subsequence q holds $\overline{\overline{q}} = \overline{\overline{\operatorname{Shift}^i q}}$.

(58) For every finite sequence p holds $\operatorname{dom} p = \operatorname{dom} \operatorname{Seq} \operatorname{Shift}^i p$.

(59) For every finite sequence p such that $k \in \operatorname{dom} p$ holds $(\operatorname{Sgm} \operatorname{dom} \operatorname{Shift}^i p)(k) = i + k$.

(60) For every finite sequence p such that $k \in \operatorname{dom} p$ holds $(\operatorname{Seq} \operatorname{Shift}^i p)(k) = p(k)$.

(61) For every finite sequence p holds $\operatorname{Seq} \operatorname{Shift}^i p = p$.

In the sequel p_1, p_2 are finite sequences.

One can prove the following propositions:

(62) $\operatorname{dom}(p_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} p_2) = \operatorname{Seg}(\operatorname{len} p_1 + \operatorname{len} p_2)$.

(63) For every finite sequence p_1 and for every finite subsequence p_2 such that $\operatorname{len} p_1 \leq i$ holds $\operatorname{dom} p_1$ misses $\operatorname{dom} \operatorname{Shift}^i p_2$.

(64) For all finite sequences p_1, p_2 holds $p_1 \frown p_2 = p_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} p_2$.

(65) For every finite sequence p_1 and for every finite subsequence p_2 such that $i \geq \operatorname{len} p_1$ holds p_1 misses $\operatorname{Shift}^i p_2$.

(66) $(R_1 \operatorname{concur} R_2) \operatorname{concur} R_3 = R_1 \operatorname{concur} (R_2 \operatorname{concur} R_3)$.

(67) $R_1 \operatorname{before} R_2 \subseteq R_1 \operatorname{concur} R_2$.

(68) If $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$, then $R_1 \operatorname{before} R_2 \subseteq P_1 \operatorname{before} P_2$.

(69) If $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$, then $R_1 \operatorname{concur} R_2 \subseteq P_1 \operatorname{concur} P_2$.

(70) For all finite subsequences p, q such that $q \subseteq p$ holds $\operatorname{Shift}^i q \subseteq \operatorname{Shift}^i p$.

(71) For all finite sequences p_1, p_2 holds $\operatorname{Shift}^{\operatorname{len} p_1} p_2 \subseteq p_1 \frown p_2$.

(72) If $\operatorname{dom} q_1$ misses $\operatorname{dom} q_2$, then $\operatorname{dom} \operatorname{Shift}^i q_1$ misses $\operatorname{dom} \operatorname{Shift}^i q_2$.

(73) For all finite subsequences q, q_1, q_2 such that $q = q_1 \cup q_2$ and q_1 misses q_2 holds $\operatorname{Shift}^i q_1 \cup \operatorname{Shift}^i q_2 = \operatorname{Shift}^i q$.

(74) For every finite subsequence q holds $\operatorname{dom} \operatorname{Seq} q = \operatorname{dom} \operatorname{Seq} \operatorname{Shift}^i q$.

(75) For every finite subsequence q such that $k \in \operatorname{dom} \operatorname{Seq} q$ there exists j such that $j = (\operatorname{Sgm} \operatorname{dom} q)(k)$ and $(\operatorname{Sgm} \operatorname{dom} \operatorname{Shift}^i q)(k) = i + j$.

(76) For every finite subsequence q such that $k \in \operatorname{dom} \operatorname{Seq} q$ holds $(\operatorname{Seq} \operatorname{Shift}^i q)(k) = (\operatorname{Seq} q)(k)$.

(77) For every finite subsequence q holds $\operatorname{Seq} q = \operatorname{Seq} \operatorname{Shift}^i q$.

(78) For every finite subsequence q such that $\operatorname{dom} q \subseteq \operatorname{Seg} k$ holds $\operatorname{dom} \operatorname{Shift}^i q \subseteq \operatorname{Seg}(i + k)$.

(79) Let p be a finite sequence and q_1, q_2 be finite subsequences. If $q_1 \subseteq p$, then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \operatorname{Shift}^{\operatorname{len} p} q_2$.

(80) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} q_2$ and $\operatorname{dom} \operatorname{Seq} s_1 = \operatorname{Seg}(\operatorname{len} \operatorname{Seq} q_1 + \operatorname{len} \operatorname{Seq} q_2)$.

(81) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} q_2$ and $\operatorname{dom} \operatorname{Seq} s_1 = \operatorname{Seg}(\operatorname{len} \operatorname{Seq} q_1 + \operatorname{len} \operatorname{Seq} q_2)$ and $\operatorname{Seq} s_1 = \operatorname{Seq} q_1 \cup \operatorname{Shift}^{\operatorname{len} \operatorname{Seq} q_1} \operatorname{Seq} q_2$.

(82) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} q_2$ and $(\operatorname{Seq} q_1) \frown \operatorname{Seq} q_2 = \operatorname{Seq} s_1$.

(83) $(R_1 \operatorname{concur} R_2) \operatorname{before} (P_1 \operatorname{concur} P_2) \subseteq (R_1 \operatorname{before} P_1) \operatorname{concur} (R_2 \operatorname{before} P_2)$.

Let us consider P, N and let R_1, R_2 be non empty processes in N. Note that $R_1 \operatorname{concur} R_2$ is non empty.

7. Neutral Process

Let us consider P and let N be a Petri net over P. The neutral process in N yields a non empty process in N and is defined as follows:

(Def. 16) The neutral process in $N = \{\varepsilon_N\}$.

Let us consider P, let N be a Petri net over P, and let t be an element of N. The elementary process with t yielding a non empty process in N is defined by:

(Def. 17) The elementary process with $t = \{\langle t \rangle\}$.

The following propositions are true:

(84) (The neutral process in N) $\operatorname{before} R = R$.

(85) $R \operatorname{before}$ the neutral process in $N = R$.

(86) (The neutral process in N) $\operatorname{concur} R = R$.

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