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Homomorphisms of Order Sorted Algebras[1]

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The articles [8], [11], [13], [14], [4], [5], [2], [1], [10], [7], [9], [3], [12], and [6] provide the notation and terminology for this paper.

In this paper R denotes a non empty poset and S_1 denotes an order sorted signature.

Let us consider R and let F be a many sorted function indexed by the carrier of R. We say that F is order-sorted if and only if:

(Def. 1) For all elements s_1, s_2 of R such that $s_1 \leq s_2$ and for every set a_1 such that $a_1 \in \operatorname{dom} F(s_1)$ holds $a_1 \in \operatorname{dom} F(s_2)$ and $F(s_1)(a_1) = F(s_2)(a_1)$.

Next we state four propositions:

(2)[1] Let F be a many sorted function indexed by the carrier of R. Suppose F is order-sorted. Let s_1, s_2 be elements of R. If $s_1 \leq s_2$, then $\operatorname{dom} F(s_1) \subseteq \operatorname{dom} F(s_2)$ and $F(s_1) \subseteq F(s_2)$.

(3) Let A be an order sorted set of R, B be a non-empty order sorted set of R, and F be a many sorted function from A into B. Then F is order-sorted if and only if for all elements s_1, s_2 of R such that $s_1 \leq s_2$ and for every set a_1 such that $a_1 \in A(s_1)$ holds $F(s_1)(a_1) = F(s_2)(a_1)$.

(4) Let F be a many sorted function indexed by the carrier of R. Suppose F is order-sorted. Let w_1, w_2 be elements of (the carrier of R)*. If $w_1 \leq w_2$, then $F^\#(w_1) \subseteq F^\#(w_2)$.

(5) For every order sorted set A of R holds id_A is order-sorted.

Let us consider R and let A be an order sorted set of R. Observe that id_A is order-sorted.

The following propositions are true:

(6) Let A be an order sorted set of R, B, C be non-empty order sorted sets of R, F be a many sorted function from A into B, and G be a many sorted function from B into C. If F is order-sorted and G is order-sorted, then $G \circ F$ is order-sorted.

(7) Let A, B be order sorted sets of R and F be a many sorted function from A into B. If F is "1-1", onto, and order-sorted, then F^{-1} is order-sorted.

(8) Let A be an order sorted set of R and F be a many sorted function indexed by the carrier of R. If F is order-sorted, then $F^\circ A$ is an order sorted set of R.

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[1] The proposition (1) has been removed.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1. We say that U_1 and U_2 are os-isomorphic if and only if:

(Def. 2) There exists a many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2 and order-sorted.

One can prove the following two propositions:

(9) For every order sorted algebra U_1 of S_1 holds U_1 and U_1 are os-isomorphic.

(10) Let U_1, U_2 be non-empty order sorted algebras of S_1. If U_1 and U_2 are os-isomorphic, then U_2 and U_1 are os-isomorphic.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1. Let us note that the predicate U_1 and U_2 are os-isomorphic is reflexive.

Let us consider S_1 and let U_1, U_2 be non-empty order sorted algebras of S_1. Let us note that the predicate U_1 and U_2 are os-isomorphic is symmetric.

One can prove the following propositions:

(11) Let U_1, U_2, U_3 be non-empty order sorted algebras of S_1. Suppose U_1 and U_2 are os-isomorphic and U_2 and U_3 are os-isomorphic. Then U_1 and U_3 are os-isomorphic.

(12) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted and a homomorphism of U_1 into U_2. Then $\operatorname{Im} F$ is order-sorted.

(13) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted. Let o_1, o_2 be operation symbols of S_1. Suppose $o_1 \leq o_2$. Let x be an element of $\operatorname{Args}(o_1, U_1)$ and x_1 be an element of $\operatorname{Args}(o_2, U_1)$. If $x = x_1$, then $F\#x = F\#x_1$.

(14) Let U_1 be a monotone non-empty order sorted algebra of S_1, U_2 be a non-empty order sorted algebra of S_1, and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted and a homomorphism of U_1 into U_2. Then $\operatorname{Im} F$ is order-sorted and $\operatorname{Im} F$ is a monotone order sorted algebra of S_1.

(15) For every monotone order sorted algebra U_1 of S_1 holds every OSSubAlgebra of U_1 is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1. Observe that there exists an OSSubAlgebra of U_1 which is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1. Observe that every OSSubAlgebra of U_1 is monotone.

Next we state two propositions:

(16) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function G from U_1 into $\operatorname{Im} F$ such that $F = G$ and G is order-sorted and an epimorphism of U_1 onto $\operatorname{Im} F$.

(17) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function F_1 from U_1 into $\operatorname{Im} F$ and there exists a many sorted function F_2 from $\operatorname{Im} F$ into U_2 such that

(i) F_1 is an epimorphism of U_1 onto $\operatorname{Im} F$,

(ii) F_2 is a monomorphism of $\operatorname{Im} F$ into U_2,

(iii) $F = F_2 \circ F_1$,

(iv) F_1 is order-sorted, and

(v) F_2 is order-sorted.

Let us consider S_1 and let U_1 be an order sorted algebra of S_1. One can verify that $\langle$the sorts of U_1, the characteristics of $U_1\rangle$ is order-sorted.

We now state two propositions:

(18) Let U_1 be an order sorted algebra of S_1. Then U_1 is monotone if and only if $\langle$the sorts of U_1, the characteristics of $U_1\rangle$ is monotone.

(19) Let U_1, U_2 be strict non-empty order sorted algebras of S_1. Suppose U_1 and U_2 are os-isomorphic. Then U_1 is monotone if and only if U_2 is monotone.

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