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Subalgebras of an Order Sorted Algebra. Lattice of Subalgebras[1]

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The articles [8], [5], [12], [14], [4], [7], [15], [3], [1], [6], [9], [10], [11], [2], and [13] provide the notation and terminology for this paper.

1. Auxiliary Facts about Order Sorted Sets

In this paper x is a set and R is a non empty poset.

One can prove the following two propositions:

(1) For all order sorted sets X, Y of R holds $X \cap Y$ is an order sorted set of R.

(2) For all order sorted sets X, Y of R holds $X \cup Y$ is an order sorted set of R.

Let R be a non empty poset and let M be an order sorted set of R. A many sorted subset indexed by M is said to be an Order sorted subset of M if:

(Def. 1) It is an order sorted set of R.

Let R be a non empty poset and let M be a non-empty order sorted set of R. One can verify that there exists an Order sorted subset of M which is non-empty.

2. Constants of an Order Sorted Algebra

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. A many sorted subset indexed by the sorts of U_0 is said to be an OSSubset of U_0 if:

(Def. 2) It is an order sorted set of S.

Let S be an order sorted signature. One can verify that there exists an order sorted algebra of S which is monotone, strict, and non-empty.

Let S be an order sorted signature and let U_0 be a non-empty order sorted algebra of S. Note that there exists an OSSubset of U_0 which is non-empty.

One can prove the following proposition

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(3) For every non void strict non empty many sorted signature S_0 with constant operations holds OSSign S_0 has constant operations.

One can check that there exists an order sorted signature which is strict and has constant operations.

3. Subalgebras of an Order Sorted Algebra

The following proposition is true

(4) Let S be an order sorted signature and U_0 be an order sorted algebra of S. Then $\langle$the sorts of U_0, the characteristics of $U_0\rangle$ is order-sorted.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. Note that there exists a subalgebra of U_0 which is order-sorted.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. An OSSubAlgebra of U_0 is an order-sorted subalgebra of U_0.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. One can verify that there exists an OSSubAlgebra of U_0 which is strict.

Let S be an order sorted signature and let U_0 be a non-empty order sorted algebra of S. One can check that there exists an OSSubAlgebra of U_0 which is non-empty and strict.

Next we state the proposition

(5) Let S be an order sorted signature, U_0 be an order sorted algebra of S, and U_1 be an algebra over S. Then U_1 is an OSSubAlgebra of U_0 if and only if the following conditions are satisfied:

(i) the sorts of U_1 are an OSSubset of U_0, and

(ii) for every OSSubset B of U_0 such that $B =$ the sorts of U_1 holds B is operations closed and the characteristics of $U_1 = \mathrm{Opers}(U_0, B)$.

We adopt the following rules: S_1 is an order sorted signature, O_0 is an order sorted algebra of S_1, and s, s_1, s_2 are sort symbols of S_1.

Let us consider S_1, O_0, s. The functor OSConstants(O_0, s) yielding a subset of (the sorts of $O_0)(s)$ is defined by:

(Def. 3) $\mathrm{OSConstants}(O_0, s) = \bigcup\{\mathrm{Constants}(O_0, s_2) : s_2 \leq s\}$.

One can prove the following proposition

(11)[1] $\mathrm{Constants}(O_0, s) \subseteq \mathrm{OSConstants}(O_0, s)$.

Let us consider S_1 and let M be a many sorted set indexed by the carrier of S_1. The functor OSCl M yielding an order sorted set of S_1 is defined by:

(Def. 4) For every sort symbol s of S_1 holds $(\mathrm{OSCl}\, M)(s) = \bigcup\{M(s_1) : s_1 \leq s\}$.

The following propositions are true:

(12) For every many sorted set M indexed by the carrier of S_1 holds $M \subseteq \mathrm{OSCl}\, M$.

(13) Let M be a many sorted set indexed by the carrier of S_1 and A be an order sorted set of S_1. If $M \subseteq A$, then $\mathrm{OSCl}\, M \subseteq A$.

(14) For every order sorted signature S and for every order sorted set X of S holds $\mathrm{OSCl}\, X = X$.

Let us consider S_1, O_0. The functor OSConstants O_0 yielding an OSSubset of O_0 is defined by:

(Def. 5) For every sort symbol s of S_1 holds $(\mathrm{OSConstants}\, O_0)(s) = \mathrm{OSConstants}(O_0, s)$.

[1] The propositions (6)–(10) have been removed.

We now state several propositions:

(15) Constants$(O_0) \subseteq$ OSConstants O_0.

(16) For every OSSubset A of O_0 such that Constants$(O_0) \subseteq A$ holds OSConstants $O_0 \subseteq A$.

(17) For every OSSubset A of O_0 holds OSConstants $O_0 =$ OSCl Constants(O_0).

(18) For every OSSubAlgebra O_1 of O_0 holds OSConstants O_0 is an OSSubset of O_1.

(19) Let S be an order sorted signature with constant operations, O_0 be a non-empty order sorted algebra of S, and O_1 be a non-empty OSSubAlgebra of O_0. Then OSConstants O_0 is a non-empty OSSubset of O_1.

4. Order Sorted Subsets of an Order Sorted Algebra

The following proposition is true

(20) Let I be a set, M be a many sorted set indexed by I, and x be a set. Then x is a many sorted subset indexed by M if and only if $x \in \prod(2^M)$.

Let R be a non empty poset and let M be an order sorted set of R. The functor OSbool M yielding a set is defined by:

(Def. 6) For every set x holds $x \in$ OSbool M iff x is an Order sorted subset of M.

Let S be an order sorted signature, let U_0 be an order sorted algebra of S, and let A be an OSSubset of U_0. The functor OSSubSort$A$ yielding a set is defined by:

(Def. 7) OSSubSort$A = \{x; x$ ranges over elements of SubSorts(A): x is an order sorted set of $S\}$.

One can prove the following propositions:

(21) For every OSSubset A of O_0 holds OSSubSort$A \subseteq$ SubSorts(A).

(22) For every OSSubset A of O_0 holds the sorts of $O_0 \in$ OSSubSort$A$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. One can verify that OSSubSort$A$ is non empty.

Let us consider S_1, O_0. The functor OSSubSort O_0 yielding a set is defined by:

(Def. 8) OSSubSort $O_0 = \{x; x$ ranges over elements of SubSorts(O_0): x is an order sorted set of $S_1\}$.

Next we state the proposition

(23) For every OSSubset A of O_0 holds OSSubSort$A \subseteq$ OSSubSort O_0.

Let us consider S_1, O_0. Observe that OSSubSort O_0 is non empty.

Let us consider S_1, O_0 and let e be an element of OSSubSort O_0. The functor ${}^@e$ yielding an OSSubset of O_0 is defined as follows:

(Def. 9) ${}^@e = e$.

We now state two propositions:

(24) For all OSSubsets A, B of O_0 holds $B \in$ OSSubSort$A$ iff B is operations closed and OSConstants $O_0 \subseteq B$ and $A \subseteq B$.

(25) For every OSSubset B of O_0 holds $B \in$ OSSubSort O_0 iff B is operations closed.

Let us consider S_1, O_0, let A be an OSSubset of O_0, and let s be an element of S_1. The functor OSSubSort(A, s) yielding a set is defined by:

(Def. 10) For every set x holds $x \in \mathrm{OSSubSort}(A, s)$ iff there exists an OSSubset B of O_0 such that $B \in \mathrm{OSSubSort}A$ and $x = B(s)$.

We now state three propositions:

(26) For every OSSubset A of O_0 and for all sort symbols s_1, s_2 of S_1 such that $s_1 \leq s_2$ holds $\mathrm{OSSubSort}(A, s_2)$ is coarser than $\mathrm{OSSubSort}(A, s_1)$.

(27) For every OSSubset A of O_0 and for every sort symbol s of S_1 holds $\mathrm{OSSubSort}(A, s) \subseteq \mathrm{SubSort}(A, s)$.

(28) For every OSSubset A of O_0 and for every sort symbol s of S_1 holds (the sorts of $O_0)(s) \in \mathrm{OSSubSort}(A, s)$.

Let us consider S_1, O_0, let A be an OSSubset of O_0, and let s be a sort symbol of S_1. One can verify that $\mathrm{OSSubSort}(A, s)$ is non empty.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. The functor OSMSubSort$A$ yielding an OSSubset of O_0 is defined by:

(Def. 11) For every sort symbol s of S_1 holds $(\mathrm{OSMSubSort}A)(s) = \bigcap \mathrm{OSSubSort}(A, s)$.

Let us consider S_1, O_0. Note that there exists an OSSubset of O_0 which is operations closed.

We now state several propositions:

(29) For every OSSubset A of O_0 holds $\mathrm{OSConstants}\, O_0 \cup A \subseteq \mathrm{OSMSubSort}A$.

(30) For every OSSubset A of O_0 such that $\mathrm{OSConstants}\, O_0 \cup A$ is non-empty holds OSMSubSort$A$ is non-empty.

(31) Let o be an operation symbol of S_1, A be an OSSubset of O_0, and B be an OSSubset of O_0. If $B \in \mathrm{OSSubSort}A$, then $((\mathrm{OSMSubSort}A)^{\#} \cdot \text{the arity of } S_1)(o) \subseteq (B^{\#} \cdot \text{the arity of } S_1)(o)$.

(32) Let o be an operation symbol of S_1, A be an OSSubset of O_0, and B be an OSSubset of O_0. Suppose $B \in \mathrm{OSSubSort}A$. Then $\mathrm{rng}(\mathrm{Den}(o, O_0) \restriction ((\mathrm{OSMSubSort}A)^{\#} \cdot \text{the arity of } S_1)(o)) \subseteq (B \cdot \text{the result sort of } S_1)(o)$.

(33) Let o be an operation symbol of S_1 and A be an OSSubset of O_0. Then $\mathrm{rng}(\mathrm{Den}(o, O_0) \restriction ((\mathrm{OSMSubSort}A)^{\#} \cdot \text{the arity of } S_1)(o)) \subseteq (\mathrm{OSMSubSort}A \cdot \text{the result sort of } S_1)(o)$.

(34) For every OSSubset A of O_0 holds OSMSubSort$A$ is operations closed and $A \subseteq \mathrm{OSMSubSort}A$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. One can verify that OSMSubSort$A$ is operations closed.

5. Operations on Subalgebras of an Order Sorted Algebra

Let us consider S_1, O_0 and let A be an operations closed OSSubset of O_0. One can verify that $O_0 \restriction A$ is order-sorted.

Let us consider S_1, O_0 and let O_1, O_2 be OSSubAlgebras of O_0. One can check that $O_1 \cap O_2$ is order-sorted.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. The functor OSGen$A$ yields a strict OSSubAlgebra of O_0 and is defined by the conditions (Def. 13).

(Def. 13)[2] (i) A is an OSSubset of OSGen$A$, and

(ii) for every OSSubAlgebra O_1 of O_0 such that A is an OSSubset of O_1 holds OSGen$A$ is an OSSubAlgebra of O_1.

One can prove the following propositions:

[2] The definition (Def. 12) has been removed.

(35) For every OSSubset A of O_0 holds $\mathrm{OSGen}A = O_0 \restriction \mathrm{OSMSubSort}A$ and the sorts of $\mathrm{OSGen}A = \mathrm{OSMSubSort}A$.

(36) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, and A be a subset of U_0. Then $\mathrm{Gen}(A) = U_0 \restriction \mathrm{MSSubSort}(A)$ and the sorts of $\mathrm{Gen}(A) = \mathrm{MSSubSort}(A)$.

(37) For every OSSubset A of O_0 holds the sorts of $\mathrm{Gen}(A) \subseteq$ the sorts of $\mathrm{OSGen}A$.

(38) For every OSSubset A of O_0 holds $\mathrm{Gen}(A)$ is a subalgebra of $\mathrm{OSGen}A$.

(39) Let O_0 be a strict order sorted algebra of S_1 and B be an OSSubset of O_0. If $B =$ the sorts of O_0, then $\mathrm{OSGen}B = O_0$.

(40) For every strict OSSubAlgebra O_1 of O_0 and for every OSSubset B of O_0 such that $B =$ the sorts of O_1 holds $\mathrm{OSGen}B = O_1$.

(41) For every non-empty order sorted algebra U_0 of S_1 and for every OSSubAlgebra U_1 of U_0 holds $\mathrm{OSGen}\,\mathrm{OSConstants}\,U_0 \cap U_1 = \mathrm{OSGen}\,\mathrm{OSConstants}\,U_0$.

Let us consider S_1, let U_0 be a non-empty order sorted algebra of S_1, and let U_1, U_2 be OSSubAlgebras of U_0. The functor $U_1 \sqcup_{os} U_2$ yielding a strict OSSubAlgebra of U_0 is defined as follows:

(Def. 14) For every OSSubset A of U_0 such that $A =$ (the sorts of U_1) $\cup$ (the sorts of U_2) holds $U_1 \sqcup_{os} U_2 = \mathrm{OSGen}A$.

One can prove the following propositions:

(42) Let U_0 be a non-empty order sorted algebra of S_1, U_1 be an OSSubAlgebra of U_0, and A, B be OSSubsets of U_0. If $B = A \cup$ the sorts of U_1, then $\mathrm{OSGen}A \sqcup_{os} U_1 = \mathrm{OSGen}B$.

(43) Let U_0 be a non-empty order sorted algebra of S_1, U_1 be an OSSubAlgebra of U_0, and B be an OSSubset of U_0. If $B =$ the sorts of U_0, then $\mathrm{OSGen}B \sqcup_{os} U_1 = \mathrm{OSGen}B$.

(44) For every non-empty order sorted algebra U_0 of S_1 and for all OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \sqcup_{os} U_2 = U_2 \sqcup_{os} U_1$.

(45) For every non-empty order sorted algebra U_0 of S_1 and for all strict OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \cap (U_1 \sqcup_{os} U_2) = U_1$.

(46) For every non-empty order sorted algebra U_0 of S_1 and for all strict OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \cap U_2 \sqcup_{os} U_2 = U_2$.

6. The Lattice of Subalgebras of an Order Sorted Algebra

Let us consider S_1, O_0. The functor $\mathrm{OSSub}\,O_0$ yields a set and is defined as follows:

(Def. 15) For every x holds $x \in \mathrm{OSSub}\,O_0$ iff x is a strict OSSubAlgebra of O_0.

One can prove the following proposition

(47) $\mathrm{OSSub}\,O_0 \subseteq \mathrm{Subalgebras}(O_0)$.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. Observe that $\mathrm{OSSub}\,U_0$ is non empty.

Let us consider S_1, O_0. Then $\mathrm{OSSub}\,O_0$ is a subset of $\mathrm{Subalgebras}(O_0)$.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor $\mathrm{OSAlgJoin}\,U_0$ yielding a binary operation on $\mathrm{OSSub}\,U_0$ is defined by:

(Def. 16) For all elements x, y of $\mathrm{OSSub}\,U_0$ and for all strict OSSubAlgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $(\mathrm{OSAlgJoin}\,U_0)(x, y) = U_1 \sqcup_{os} U_2$.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor $\mathrm{OSAlgMeet}\,U_0$ yields a binary operation on $\mathrm{OSSub}\,U_0$ and is defined as follows:

(Def. 17) For all elements x, y of OSSub U_0 and for all strict OSSubAlgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $(\mathrm{OSAlgMeet}\,U_0)(x, y) = U_1 \cap U_2$.

Next we state the proposition

(48) For every non-empty order sorted algebra U_0 of S_1 and for all elements x, y of OSSub U_0 holds $(\mathrm{OSAlgMeet}\,U_0)(x, y) = (\mathrm{MSAlgMeet}(U_0))(x, y)$.

In the sequel U_0 is a non-empty order sorted algebra of S_1.
Next we state four propositions:

(49) OSAlgJoin U_0 is commutative.

(50) OSAlgJoin U_0 is associative.

(51) OSAlgMeet U_0 is commutative.

(52) OSAlgMeet U_0 is associative.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor OSSubAlLattice U_0 yields a strict lattice and is defined as follows:

(Def. 18) $\mathrm{OSSubAlLattice}\,U_0 = \langle \mathrm{OSSub}\,U_0, \mathrm{OSAlgJoin}\,U_0, \mathrm{OSAlgMeet}\,U_0 \rangle$.

Next we state the proposition

(53) For every non-empty order sorted algebra U_0 of S_1 holds OSSubAlLattice U_0 is bounded.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. Observe that OSSubAlLattice U_0 is bounded.

We now state three propositions:

(54) For every non-empty order sorted algebra U_0 of S_1 holds $\perp_{\mathrm{OSSubAlLattice}\,U_0} = \mathrm{OSGen}\,\mathrm{OSConstants}\,U_0$.

(55) Let U_0 be a non-empty order sorted algebra of S_1 and B be an OSSubset of U_0. If $B =$ the sorts of U_0, then $\top_{\mathrm{OSSubAlLattice}\,U_0} = \mathrm{OSGen}\,B$.

(56) For every strict non-empty order sorted algebra U_0 of S_1 holds $\top_{\mathrm{OSSubAlLattice}\,U_0} = U_0$.

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