JOURNAL OF FORMALIZED MATHEMATICS
Volume 14, Released 2002, Published 2003
Inst. of Computer Science, Univ. of Białystok

Free Order Sorted Universal Algebra[1]

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Summary. Free Order Sorted Universal Algebra — the general construction for any locally directed signatures.

MML Identifier: OSAFREE.

WWW: http://mizar.org/JFM/Vol14/osafree.html

The articles [22], [13], [28], [33], [34], [10], [23], [12], [11], [7], [14], [35], [4], [19], [2], [21], [27], [15], [5], [3], [6], [1], [8], [26], [24], [18], [25], [9], [16], [17], [30], [32], [29], [31], and [20] provide the notation and terminology for this paper.

1. PRELIMINARIES

In this paper S is an order sorted signature.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. A subset of U_0 is called an order sorted generator set of U_0 if:

(Def. 1) For every OSSubset O of U_0 such that $O = \mathrm{OSCl}\,\mathrm{it}$ holds the sorts of $\mathrm{OSGen}\,O =$ the sorts of U_0.

One can prove the following proposition

(1) Let S be an order sorted signature, U_0 be a strict non-empty order sorted algebra of S, and A be a subset of U_0. Then A is an order sorted generator set of U_0 if and only if for every OSSubset O of U_0 such that $O = \mathrm{OSCl}\,A$ holds $\mathrm{OSGen}\,O = U_0$.

Let us consider S, let U_0 be a monotone order sorted algebra of S, and let I_1 be an order sorted generator set of U_0. We say that I_1 is osfree if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let U_1 be a monotone non-empty order sorted algebra of S and f be a many sorted function from I_1 into the sorts of U_1. Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and order-sorted and $h \restriction I_1 = f$.

Let S be an order sorted signature and let I_1 be a monotone order sorted algebra of S. We say that I_1 is osfree if and only if:

(Def. 3) There exists an order sorted generator set of I_1 which is osfree.

[1]This work was done during author's research visit in Bialystok, funded by the CALCULEMUS grant HPRN-CT-2000-00102.

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2. Construction of Free Order Sorted Algebras for Given Signature

Let S be an order sorted signature and let X be a many sorted set indexed by S. The functor OSRELX yields a relation between $[:$the operation symbols of S, $\{$the carrier of $S\}:] \cup \bigcup$coprod(X) and $([:$the operation symbols of S, $\{$the carrier of $S\}:] \cup \bigcup$coprod$(X))^*$ and is defined by the condition (Def. 4).

(Def. 4) Let a be an element of $[:$the operation symbols of S, $\{$the carrier of $S\}:] \cup \bigcup$coprod(X) and b be an element of $([:$the operation symbols of S, $\{$the carrier of $S\}:] \cup \bigcup$coprod$(X))^*$. Then $\langle a, b\rangle \in$ OSRELX if and only if the following conditions are satisfied:

(i) $a \in [:$the operation symbols of S, $\{$the carrier of $S\}:]$, and

(ii) for every operation symbol o of S such that $\langle o,$ the carrier of $S\rangle = a$ holds len$b =$ len Arity(o) and for every set x such that $x \in$ dom$b$ holds if $b(x) \in [:$the operation symbols of S, $\{$the carrier of $S\}:]$, then for every operation symbol o_1 of S such that $\langle o_1,$ the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 \leq$ Arity$(o)_x$ and if $b(x) \in \bigcup$coprod(X), then there exists an element i of S such that $i \leq$ Arity$(o)_x$ and $b(x) \in$ coprod(i,X).

In the sequel S denotes an order sorted signature, X denotes a many sorted set indexed by S, o denotes an operation symbol of S, and b denotes an element of $([:$the operation symbols of S, $\{$the carrier of $S\}:] \cup \bigcup$coprod$(X))^*$.

We now state the proposition

(2) $\langle\langle o,$ the carrier of $S\rangle, b\rangle \in$ OSRELX if and only if the following conditions are satisfied:

(i) len$b =$ len Arity(o), and

(ii) for every set x such that $x \in$ dom$b$ holds if $b(x) \in [:$the operation symbols of S, $\{$the carrier of $S\}:]$, then for every operation symbol o_1 of S such that $\langle o_1,$ the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 \leq$ Arity$(o)_x$ and if $b(x) \in \bigcup$coprod(X), then there exists an element i of S such that $i \leq$ Arity$(o)_x$ and $b(x) \in$ coprod(i,X).

Let S be an order sorted signature and let X be a many sorted set indexed by S. The functor DTConOSAX yields a tree construction structure and is defined by:

(Def. 5) DTConOSA$X = \langle [:$the operation symbols of S, $\{$the carrier of $S\}:] \cup \bigcup$coprod(X), OSREL$X\rangle$.

Let S be an order sorted signature and let X be a many sorted set indexed by S. Note that DTConOSAX is strict and non empty.

The following proposition is true

(3) Let S be an order sorted signature and X be a non-empty many sorted set indexed by S. Then the nonterminals of DTConOSA$X = [:$the operation symbols of S, $\{$the carrier of $S\}:]$ and the terminals of DTConOSA$X = \bigcup$coprod(X).

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. Note that DTConOSAX has terminals, nonterminals, and useful nonterminals.

We now state the proposition

(4) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and t be a set. Then $t \in$ the terminals of DTConOSAX if and only if there exists an element s of S and there exists a set x such that $x \in X(s)$ and $t = \langle x, s\rangle$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. The functor OSSym(o,X) yields a symbol of DTConOSAX and is defined as follows:

(Def. 6) OSSym$(o,X) = \langle o,$ the carrier of $S\rangle$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. The functor ParsedTerms(X,s) yields a subset of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ and is defined by the condition (Def. 7).

(Def. 7) $\mathrm{ParsedTerms}(X,s) = \{a; a$ ranges over elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$: $\bigvee_{s_1 : \text{element of } S} \bigvee_{x : \text{set}} (s_1 \leq s \wedge x \in X(s_1) \wedge a = \text{the root tree of } \langle x, s_1 \rangle) \vee \bigvee_{o : \text{operation symbol of } S} (\langle o, \text{the carrier of } S \rangle = a(\emptyset) \wedge \text{the result sort of } o \leq s)\}$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. Observe that ParsedTerms(X,s) is non empty.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor ParsedTerms$X$ yields an order sorted set of S and is defined by:

(Def. 8) For every element s of S holds $(\mathrm{ParsedTerms}\,X)(s) = \mathrm{ParsedTerms}(X,s)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. Observe that ParsedTerms$X$ is non-empty.

Next we state four propositions:

(5) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a set. Suppose $x \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$. Then x is a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$.

(6) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and p be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $p \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$ if and only if $\mathrm{dom}\,p = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number n such that $n \in \mathrm{dom}\,p$ holds $p(n) \in \mathrm{ParsedTerms}(X, \mathrm{Arity}(o)_n)$.

(7) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and p be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $\mathrm{OSSym}(o,X) \Rightarrow$ the roots of p if and only if $p \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$.

(8) For every order sorted signature S and for every non-empty many sorted set X indexed by S holds $\bigcup \mathrm{rng}\,\mathrm{ParsedTerms}\,X = \mathrm{TS}(\mathrm{DTConOSA}\,X)$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. The functor PTDenOp(o,X) yields a function from $((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$ into $(\mathrm{ParsedTerms}\,X \cdot \text{the result sort of } S)(o)$ and is defined by:

(Def. 9) For every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\mathrm{OSSym}(o,X) \Rightarrow$ the roots of p holds $(\mathrm{PTDenOp}(o,X))(p) = \mathrm{OSSym}(o,X)\text{-tree}(p)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTOper$X$ yields a many sorted function from $(\mathrm{ParsedTerms}\,X)^{\#} \cdot$ the arity of S into $\mathrm{ParsedTerms}\,X \cdot$ the result sort of S and is defined as follows:

(Def. 10) For every operation symbol o of S holds $(\mathrm{PTOper}\,X)(o) = \mathrm{PTDenOp}(o,X)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor ParsedTermsOSAX yields an order sorted algebra of S and is defined by:

(Def. 11) $\mathrm{ParsedTermsOSA}\,X = \langle \mathrm{ParsedTerms}\,X, \mathrm{PTOper}\,X \rangle$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. Note that ParsedTermsOSAX is strict and non-empty.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. Then $\mathrm{OSSym}(o,X)$ is a nonterminal of DTConOSAX.

One can prove the following propositions:

(9) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of S. Then (the sorts of ParsedTermsOSAX)$(s) = \{a; a$ ranges over elements of TS(DTConOSAX): $\bigvee_{s_1 : \text{element of } S} \bigvee_{x : \text{set}} (s_1 \leq s \wedge x \in X(s_1) \wedge a =$ the root tree of $\langle x, s_1 \rangle) \vee \bigvee_{o : \text{operation symbol of } S} (\langle o$, the carrier of $S\rangle = a(\emptyset) \wedge$ the result sort of $o \leq s)\}$.

(10) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s, s_1 be elements of S, and x be a set. Suppose $x \in X(s)$. Then

(i) the root tree of $\langle x, s\rangle$ is an element of TS(DTConOSAX),

(ii) for every set z holds $\langle z$, the carrier of $S\rangle \neq$ (the root tree of $\langle x, s\rangle)(\emptyset)$, and

(iii) the root tree of $\langle x, s\rangle \in$ (the sorts of ParsedTermsOSAX)(s_1) iff $s \leq s_1$.

(11) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, t be an element of TS(DTConOSAX), and o be an operation symbol of S. Suppose $t(\emptyset) = \langle o$, the carrier of $S\rangle$. Then

(i) there exists a subtree sequence p joinable by OSSym(o,X) such that $t =$ OSSym(o,X)-tree(p) and OSSym$(o,X) \Rightarrow$ the roots of p and $p \in$ Args$(o,$ParsedTermsOSA$X)$ and $t = ($Den$(o,$ParsedTermsOSA$X))(p)$,

(ii) for every element s_2 of S and for every set x holds $t \neq$ the root tree of $\langle x, s_2\rangle$, and

(iii) for every element s_1 of S holds $t \in$ (the sorts of ParsedTermsOSAX)(s_1) iff the result sort of $o \leq s_1$.

(12) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, n_1 be a symbol of DTConOSAX, and t_1 be a finite sequence of elements of TS(DTConOSAX). Suppose $n_1 \Rightarrow$ the roots of t_1. Then

(i) $n_1 \in$ the nonterminals of DTConOSAX,

(ii) n_1-tree$(t_1) \in$ TS(DTConOSAX), and

(iii) there exists an operation symbol o of S such that $n_1 = \langle o$, the carrier of $S\rangle$ and $t_1 \in$ Args$(o,$ParsedTermsOSA$X)$ and n_1-tree$(t_1) = ($Den$(o,$ParsedTermsOSA$X))(t_1)$ and for every element s_1 of S holds n_1-tree$(t_1) \in$ (the sorts of ParsedTermsOSAX)(s_1) iff the result sort of $o \leq s_1$.

(13) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a finite sequence. Then $x \in$ Args$(o,$ParsedTermsOSA$X)$ if and only if the following conditions are satisfied:

(i) x is a finite sequence of elements of TS(DTConOSAX), and

(ii) OSSym$(o,X) \Rightarrow$ the roots of x.

(14) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of TS(DTConOSAX). Then there exists a sort symbol s of S such that $t \in$ (the sorts of ParsedTermsOSAX)(s) and for every element s_1 of S such that $t \in$ (the sorts of ParsedTermsOSAX)(s_1) holds $s \leq s_1$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be an element of TS(DTConOSAX). The functor LeastSort$t$ yields a sort symbol of S and is defined by:

(Def. 12) $t \in$ (the sorts of ParsedTermsOSAX)(LeastSort$t$) and for every element s_1 of S such that $t \in$ (the sorts of ParsedTermsOSAX)(s_1) holds LeastSort$t \leq s_1$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. An element of A is an element of $\bigcup$(the sorts of A).

One can prove the following propositions:

(15) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and x be a set. Then x is an element of ParsedTermsOSAX if and only if x is an element of TS(DTConOSAX).

(16) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of S, and x be a set. If $x \in$ (the sorts of ParsedTermsOSAX)(s), then x is an element of TS(DTConOSAX).

(17) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of S, and x be a set. Suppose $x \in X(s)$. Let t be an element of TS(DTConOSAX). If $t =$ the root tree of $\langle x, s\rangle$, then LeastSort$t = s$.

(18) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, x be an element of Args(o,ParsedTermsOSAX), and t be an element of TS(DTConOSAX). If $t =$ (Den(o,ParsedTermsOSAX))(x), then LeastSort$t =$ the result sort of o.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o_2 be an operation symbol of S. Note that Args(o_2,ParsedTermsOSAX) is non empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let x be a finite sequence of elements of TS(DTConOSAX). The functor LeastSorts$x$ yielding an element of (the carrier of S)* is defined by:

(Def. 14)[1] dom LeastSorts$x =$ dom$x$ and for every natural number y such that $y \in$ dom$x$ there exists an element t of TS(DTConOSAX) such that $t = x(y)$ and (LeastSorts$x$)$(y) =$ LeastSort$t$.

One can prove the following proposition

(19) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a finite sequence of elements of TS(DTConOSAX). Then LeastSorts$x \leq$ Arity(o) if and only if $x \in$ Args(o,ParsedTermsOSAX).

One can verify that there exists a monotone order sorted signature which is locally directed and regular.

Let S be a locally directed regular monotone order sorted signature, let X be a non-empty many sorted set indexed by S, let o be an operation symbol of S, and let x be a finite sequence of elements of TS(DTConOSAX). Let us assume that OSSym(LBound(o,LeastSorts$x$),X) $\Rightarrow$ the roots of x. The functor $\pi_x o$ yielding an element of TS(DTConOSAX) is defined as follows:

(Def. 15) $\pi_x o =$ OSSym(LBound(o,LeastSorts$x$),X)-tree(x).

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of DTConOSAX. Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor $^{@}(X,t)$ yielding an operation symbol of S is defined by:

(Def. 16) $\langle ^{@}(X,t)$, the carrier of $S\rangle = t$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of DTConOSAX. Let us assume that $t \in$ the terminals of DTConOSAX. The functor $\prod t$ yielding an element of TS(DTConOSAX) is defined by:

(Def. 17) $\prod t =$ the root tree of t.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor LCongruence$X$ yields a monotone order sorted congruence of ParsedTermsOSAX and is defined by:

(Def. 18) For every monotone order sorted congruence R of ParsedTermsOSAX holds LCongruence$X \subseteq R$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor FreeOSAX yielding a strict non-empty monotone order sorted algebra of S is defined by:

[1] The definition (Def. 13) has been removed.

(Def. 19) $\mathrm{FreeOSA}\,X = \mathrm{QuotOSAlg}(\mathrm{ParsedTermsOSA}\,X, \mathrm{LCongruence}\,X)$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of DTConOSAX. The functor ${}^{@}t$ yielding a subset of $[:\mathrm{TS}(\mathrm{DTConOSA}\,X)$, the carrier of $S:]$ is defined by the condition (Def. 20).

(Def. 20) ${}^{@}t = \{\langle$the root tree of $t,\ s_1\rangle; s_1$ ranges over elements of S: $\bigvee_{s\,:\,\text{element of } S} \bigvee_{x\,:\,\text{set}} (x \in X(s) \wedge t = \langle x, s\rangle \wedge s \leq s_1)\}$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, let n_1 be a symbol of DTConOSAX, and let x be a finite sequence of elements of $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\,\text{the carrier of } S:]}$. The functor ${}^{@}(n_1,x)$ yielding a subset of $[:\mathrm{TS}(\mathrm{DTConOSA}\,X)$, the carrier of $S:]$ is defined by the condition (Def. 21).

(Def. 21) ${}^{@}(n_1,x) = \{\langle(\mathrm{Den}(o_2, \mathrm{ParsedTermsOSA}\,X))(x_2), s_3\rangle; o_2$ ranges over operation symbols of S, x_2 ranges over elements of $\mathrm{Args}(o_2, \mathrm{ParsedTermsOSA}\,X)$, s_3 ranges over elements of S: $\bigvee_{o_1\,:\,\text{operation symbol of } S} (n_1 = \langle o_1,$ the carrier of $S\rangle \wedge o_1 \cong o_2 \wedge \mathrm{len}\,\mathrm{Arity}(o_1) = \mathrm{len}\,\mathrm{Arity}(o_2) \wedge$ the result sort of $o_1 \leq s_3 \wedge$ the result sort of $o_2 \leq s_3) \wedge \bigvee_{w_3\,:\,\text{element of (the carrier of } S)^*} (\mathrm{dom}\,w_3 = \mathrm{dom}\,x \wedge \bigwedge_{y\,:\,\text{natural number}} (y \in \mathrm{dom}\,x \Rightarrow \langle x_2(y), (w_3)_y\rangle \in x(y)))\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTClasses$X$ yielding a function from $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ into $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\,\text{the carrier of } S:]}$ is defined by the conditions (Def. 22).

(Def. 22)(i) For every symbol t of DTConOSAX such that $t \in$ the terminals of DTConOSAX holds $(\mathrm{PTClasses}\,X)($the root tree of $t) = {}^{@}t$, and

(ii) for every symbol n_1 of DTConOSAX and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $n_1 \Rightarrow$ the roots of t_1 holds $(\mathrm{PTClasses}\,X)(n_1\text{-tree}(t_1)) = {}^{@}(n_1, \mathrm{PTClasses}\,X \cdot t_1)$.

Next we state four propositions:

(20) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then

(i) for every element s of S holds $t \in$ (the sorts of ParsedTermsOSA$X)(s)$ iff $\langle t, s\rangle \in (\mathrm{PTClasses}\,X)(t)$, and

(ii) for every element s of S and for every element y of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\langle y, s\rangle \in (\mathrm{PTClasses}\,X)(t)$ holds $\langle t, s\rangle \in (\mathrm{PTClasses}\,X)(y)$.

(21) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and s be an element of S. If there exists an element y of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\langle y, s\rangle \in (\mathrm{PTClasses}\,X)(t)$, then $\langle t, s\rangle \in (\mathrm{PTClasses}\,X)(t)$.

(22) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, x, y be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and s_1, s_2 be elements of S. Suppose $s_1 \leq s_2$ and $x \in$ (the sorts of ParsedTermsOSA$X)(s_1)$ and $y \in$ (the sorts of ParsedTermsOSA$X)(s_1)$. Then $\langle y, s_1\rangle \in (\mathrm{PTClasses}\,X)(x)$ if and only if $\langle y, s_2\rangle \in (\mathrm{PTClasses}\,X)(x)$.

(23) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, x, y, z be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and s be an element of S. If $\langle y, s\rangle \in (\mathrm{PTClasses}\,X)(x)$ and $\langle z, s\rangle \in (\mathrm{PTClasses}\,X)(y)$, then $\langle x, s\rangle \in (\mathrm{PTClasses}\,X)(z)$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTCongruence$X$ yields an equivalence order sorted relation of ParsedTermsOSAX and is defined by the condition (Def. 23).

(Def. 23) Let i be a set. Suppose $i \in$ the carrier of S. Then $(\mathrm{PTCongruence}\,X)(i) = \{\langle x, y\rangle; x$ ranges over elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, y ranges over elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$: $\langle x, i\rangle \in (\mathrm{PTClasses}\,X)(y)\}$.

The following propositions are true:

(24) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and x, y, s be sets. If $\langle x, s\rangle \in (\mathrm{PTClasses}\,X)(y)$, then $x \in \mathrm{TS}(\mathrm{DTConOSA}\,X)$ and $y \in \mathrm{TS}(\mathrm{DTConOSA}\,X)$ and $s \in$ the carrier of S.

(25) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, C be a component of S, and x, y be sets. Then $\langle x, y\rangle \in \mathrm{CompClass}(\mathrm{PTCongruence}\,X, C)$ if and only if there exists an element s_1 of S such that $s_1 \in C$ and $\langle x, s_1\rangle \in (\mathrm{PTClasses}\,X)(y)$.

(26) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of S, and x be an element of (the sorts of $\mathrm{ParsedTermsOSA}\,X)(s)$. Then $\mathrm{OSClass}(\mathrm{PTCongruence}\,X, x) = \pi_1((\mathrm{PTClasses}\,X)(x))$.

(27) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and R be a many sorted relation indexed by $\mathrm{ParsedTermsOSA}\,X$. Then $R = \mathrm{PTCongruence}\,X$ if and only if the following conditions are satisfied:

(i) for all elements s_1, s_2 of S and for every set x such that $x \in X(s_1)$ holds if $s_1 \leq s_2$, then $\langle$the root tree of $\langle x, s_1\rangle$, the root tree of $\langle x, s_1\rangle\rangle \in R(s_2)$ and for every set y such that $\langle$the root tree of $\langle x, s_1\rangle$, $y\rangle \in R(s_2)$ or $\langle y$, the root tree of $\langle x, s_1\rangle\rangle \in R(s_2)$ holds $s_1 \leq s_2$ and $y =$ the root tree of $\langle x, s_1\rangle$, and

(ii) for all operation symbols o_1, o_2 of S and for every element x_1 of $\mathrm{Args}(o_1, \mathrm{ParsedTermsOSA}\,X)$ and for every element x_2 of $\mathrm{Args}(o_2, \mathrm{ParsedTermsOSA}\,X)$ and for every element s_3 of S holds $\langle(\mathrm{Den}(o_1, \mathrm{ParsedTermsOSA}\,X))(x_1), (\mathrm{Den}(o_2, \mathrm{ParsedTermsOSA}\,X))(x_2)\rangle \in R(s_3)$ iff $o_1 \cong o_2$ and $\mathrm{len}\,\mathrm{Arity}(o_1) = \mathrm{len}\,\mathrm{Arity}(o_2)$ and the result sort of $o_1 \leq s_3$ and the result sort of $o_2 \leq s_3$ and there exists an element w_3 of (the carrier of $S)^*$ such that $\mathrm{dom}\,w_3 = \mathrm{dom}\,x_1$ and for every natural number y such that $y \in \mathrm{dom}\,w_3$ holds $\langle x_1(y), x_2(y)\rangle \in R((w_3)_y)$.

(28) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{PTCongruence}\,X$ is monotone.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. Observe that $\mathrm{PTCongruence}\,X$ is monotone.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. The functor $\mathrm{PTVars}(s, X)$ yields a subset of (the sorts of $\mathrm{ParsedTermsOSA}\,X)(s)$ and is defined as follows:

(Def. 24) For every set x holds $x \in \mathrm{PTVars}(s, X)$ iff there exists a set a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s\rangle$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. Note that $\mathrm{PTVars}(s, X)$ is non empty.

Next we state the proposition

(29) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of S. Then $\mathrm{PTVars}(s, X) = \{$the root tree of t; t ranges over symbols of $\mathrm{DTConOSA}\,X$: $t \in$ the terminals of $\mathrm{DTConOSA}\,X \wedge t_{\mathbf{2}} = s\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{PTVars}\,X$ yielding a subset of $\mathrm{ParsedTermsOSA}\,X$ is defined by:

(Def. 25) For every element s of S holds $(\mathrm{PTVars}\,X)(s) = \mathrm{PTVars}(s, X)$.

One can prove the following proposition

(30) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{PTVars}\,X$ is non-empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. The functor $\mathrm{OSFreeGen}(s,X)$ yielding a subset of (the sorts of $\mathrm{FreeOSA}X)(s)$ is defined as follows:

(Def. 26) For every set x holds $x \in \mathrm{OSFreeGen}(s,X)$ iff there exists a set a such that $a \in X(s)$ and $x = (\mathrm{OSNatHom}(\mathrm{ParsedTermsOSA}X, \mathrm{LCongruence}X))(s)$(the root tree of $\langle a, s\rangle$).

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. One can verify that $\mathrm{OSFreeGen}(s,X)$ is non empty.

Next we state the proposition

(31) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of S. Then $\mathrm{OSFreeGen}(s,X) = \{(\mathrm{OSNatHom}(\mathrm{ParsedTermsOSA}X, \mathrm{LCongruence}X))(s)(\text{the root tree of } t); t \text{ ranges over symbols of } \mathrm{DTConOSA}X : t \in \text{the terminals of } \mathrm{DTConOSA}X \wedge t_{\mathbf{2}} = s\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{OSFreeGen}X$ yields an order sorted generator set of $\mathrm{FreeOSA}X$ and is defined by:

(Def. 27) For every element s of S holds $(\mathrm{OSFreeGen}X)(s) = \mathrm{OSFreeGen}(s,X)$.

The following proposition is true

(32) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{OSFreeGen}X$ is non-empty.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. One can check that $\mathrm{OSFreeGen}X$ is non-empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, let R be an order sorted congruence of $\mathrm{ParsedTermsOSA}X$, and let t be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$. The functor $\mathrm{OSClass}(R,t)$ yields an element of $\mathrm{OSClass}(R, \mathrm{LeastSort}\,t)$ and is defined as follows:

(Def. 28) For every element s of S and for every element x of (the sorts of $\mathrm{ParsedTermsOSA}X)(s)$ such that $t = x$ holds $\mathrm{OSClass}(R,t) = \mathrm{OSClass}(R,x)$.

Next we state several propositions:

(33) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R be an order sorted congruence of $\mathrm{ParsedTermsOSA}X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$. Then $t \in \mathrm{OSClass}(R,t)$.

(34) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of S, t be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$, and x, x_1 be sets. Suppose $x \in X(s)$ and $t =$ the root tree of $\langle x, s\rangle$. Then $x_1 \in \mathrm{OSClass}(\mathrm{PTCongruence}X, t)$ if and only if $x_1 = t$.

(35) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R be an order sorted congruence of $\mathrm{ParsedTermsOSA}X$, and t_2, t_3 be elements of $\mathrm{TS}(\mathrm{DTConOSA}X)$. Then $t_3 \in \mathrm{OSClass}(R,t_2)$ if and only if $\mathrm{OSClass}(R,t_2) = \mathrm{OSClass}(R,t_3)$.

(36) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R_1, R_2 be order sorted congruences of $\mathrm{ParsedTermsOSA}X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$. If $R_1 \subseteq R_2$, then $\mathrm{OSClass}(R_1,t) \subseteq \mathrm{OSClass}(R_2,t)$.

(37) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of S, t be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$, and x, x_1 be sets. Suppose $x \in X(s)$ and $t =$ the root tree of $\langle x, s\rangle$. Then $x_1 \in \mathrm{OSClass}(\mathrm{LCongruence}X, t)$ if and only if $x_1 = t$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, let A be a non-empty many sorted set indexed by the carrier of S, let F be a many sorted function from PTVars$X$ into A, and let t be a symbol of DTConOSAX. Let us assume that $t \in$ the terminals of DTConOSAX. The functor $\pi(F,A,t)$ yielding an element of $\bigcup A$ is defined as follows:

(Def. 29) For every function f such that $f = F(t_2)$ holds $\pi(F,A,t) = f$(the root tree of t).

One can prove the following proposition

(38) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, U_1 be a monotone non-empty order sorted algebra of S, and f be a many sorted function from PTVars$X$ into the sorts of U_1. Then there exists a many sorted function h from ParsedTermsOSAX into U_1 such that h is a homomorphism of ParsedTermsOSAX into U_1 and order-sorted and $h \restriction$ PTVars$X = f$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of S. The functor NHReverse(s,X) yields a function from OSFreeGen(s,X) into PTVars(s,X) and is defined by the condition (Def. 30).

(Def. 30) Let t be a symbol of DTConOSAX. Suppose (OSNatHom(ParsedTermsOSAX, LCongruence$X$))(s)(the root tree of t) $\in$ OSFreeGen(s,X). Then (NHReverse(s,X))((OSNatHom(ParsedTermsOSAX, LCongruence$X$))(s)(the root tree of t)) = the root tree of t.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor NHReverse$X$ yields a many sorted function from OSFreeGen$X$ into PTVars$X$ and is defined by:

(Def. 31) For every element s of S holds (NHReverse$X$)(s) = NHReverse(s,X).

We now state two propositions:

(39) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then OSFreeGen$X$ is osfree.

(40) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then FreeOSAX is osfree.

Let S be a locally directed order sorted signature. Observe that there exists a non-empty monotone order sorted algebra of S which is osfree and strict.

3. MINIMAL TERMS

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTMin$X$ yields a function from TS(DTConOSAX) into TS(DTConOSAX) and is defined by the conditions (Def. 32).

(Def. 32)(i) For every symbol t of DTConOSAX such that $t \in$ the terminals of DTConOSAX holds (PTMin$X$)(the root tree of t) $= \prod t$, and

(ii) for every symbol n_1 of DTConOSAX and for every finite sequence t_1 of elements of TS(DTConOSAX) such that $n_1 \Rightarrow$ the roots of t_1 holds (PTMin$X$)(n_1-tree(t_1)) $= \pi_{\mathrm{PTMin}X \cdot t_1}(^{@}(X,n_1))$.

The following propositions are true:

(41) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of TS(DTConOSAX). Then

(i) (PTMin$X$)$(t) \in$ OSClass(PTCongruence$X$, t),

(ii) LeastSort(PTMin$X$)$(t) \leq$ LeastSort$t$,

(iii) for every element s of S and for every set x such that $x \in X(s)$ and $t =$ the root tree of $\langle x, s\rangle$ holds $(\mathrm{PTMin}X)(t) = t$, and

(iv) for every operation symbol o of S and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathrm{DTConOSA}X)$ such that $\mathrm{OSSym}(o,X) \Rightarrow$ the roots of t_1 and $t = \mathrm{OSSym}(o,X)$-tree(t_1) holds $\mathrm{LeastSorts}\,\mathrm{PTMin}X \cdot t_1 \leq \mathrm{Arity}(o)$ and $\mathrm{OSSym}(o,X) \Rightarrow$ the roots of $\mathrm{PTMin}X \cdot t_1$ and $\mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,\mathrm{PTMin}X \cdot t_1), X) \Rightarrow$ the roots of $\mathrm{PTMin}X \cdot t_1$ and $(\mathrm{PTMin}X)(t) = \mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,\mathrm{PTMin}X \cdot t_1), X)$-tree$(\mathrm{PTMin}X \cdot t_1)$.

(42) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t, t_2 be elements of $\mathrm{TS}(\mathrm{DTConOSA}X)$. If $t_2 \in \mathrm{OSClass}(\mathrm{PTCongruence}X, t)$, then $(\mathrm{PTMin}X)(t_2) = (\mathrm{PTMin}X)(t)$.

(43) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t_2, t_3 be elements of $\mathrm{TS}(\mathrm{DTConOSA}X)$. Then $t_3 \in \mathrm{OSClass}(\mathrm{PTCongruence}X, t_2)$ if and only if $(\mathrm{PTMin}X)(t_3) = (\mathrm{PTMin}X)(t_2)$.

(44) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t_2 be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$. Then $(\mathrm{PTMin}X)((\mathrm{PTMin}X)(t_2)) = (\mathrm{PTMin}X)(t_2)$.

(45) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, R be a monotone equivalence order sorted relation of $\mathrm{ParsedTermsOSA}X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}X)$. Then $\langle t, (\mathrm{PTMin}X)(t)\rangle \in R(\mathrm{LeastSort}\,t)$.

(46) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and R be a monotone equivalence order sorted relation of $\mathrm{ParsedTermsOSA}X$. Then $\mathrm{PTCongruence}X \subseteq R$.

(47) Let S be a locally directed regular monotone order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{LCongruence}X = \mathrm{PTCongruence}X$.

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. An element of $\mathrm{TS}(\mathrm{DTConOSA}X)$ is called a minimal term of S, X if:

(Def. 33) $(\mathrm{PTMin}X)(\mathrm{it}) = \mathrm{it}$.

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{MinTerms}X$ yielding a subset of $\mathrm{TS}(\mathrm{DTConOSA}X)$ is defined by:

(Def. 34) $\mathrm{MinTerms}X = \mathrm{rng}\,\mathrm{PTMin}X$.

The following proposition is true

(48) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and x be a set. Then x is a minimal term of S, X if and only if $x \in \mathrm{MinTerms}X$.

ACKNOWLEDGMENTS

Thanks to Joseph Goguen, for providing me with his articles on osas, and Andrzej Trybulec, for suggesting and funding this work in Bialystok.

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Received September 19, 2002

Published January 2, 2004