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The Ordinal Numbers

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Summary. In the beginning of the article we show some consequences of the regularity axiom. In the second part we introduce the successor of a set and the notions of transitivity and connectedness wrt membership relation. Then we define ordinal numbers as transitive and connected sets, and we prove some theorems of them and of their sets. Lastly we introduce the concept of a transfinite sequence and we show transfinite induction and schemes of defining by transfinite induction.

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The articles [2], [3], and [1] provide the notation and terminology for this paper.

In this paper X, Y, Z, X_1, X_2, X_3, X_4, X_5, X_6, x are sets.

We now state several propositions:

(3)[1] $X \notin Y$ or $Y \notin Z$ or $Z \notin X$.

(4) $X_1 \notin X_2$ or $X_2 \notin X_3$ or $X_3 \notin X_4$ or $X_4 \notin X_1$.

(5) $X_1 \notin X_2$ or $X_2 \notin X_3$ or $X_3 \notin X_4$ or $X_4 \notin X_5$ or $X_5 \notin X_1$.

(6) $X_1 \notin X_2$ or $X_2 \notin X_3$ or $X_3 \notin X_4$ or $X_4 \notin X_5$ or $X_5 \notin X_6$ or $X_6 \notin X_1$.

(7) If $Y \in X$, then $X \not\subseteq Y$.

Let us consider X. The functor $\mathrm{succ}\, X$ yields a set and is defined by:

(Def. 1) $\mathrm{succ}\, X = X \cup \{X\}$.

Let us consider X. Observe that $\mathrm{succ}\, X$ is non empty.

The following four propositions are true:

(10)[2] $X \in \mathrm{succ}\, X$.

(12)[3] If $\mathrm{succ}\, X = \mathrm{succ}\, Y$, then $X = Y$.

(13) $x \in \mathrm{succ}\, X$ iff $x \in X$ or $x = X$.

(14) $X \neq \mathrm{succ}\, X$.

In the sequel a, X, Y, Z, x, y denote sets.

Let us consider X. We say that X is transitive if and only if:

[1] The propositions (1) and (2) have been removed.
[2] The propositions (8) and (9) have been removed.
[3] The proposition (11) has been removed.

(Def. 2) For every x such that $x \in X$ holds $x \subseteq X$.

We say that X is connected if and only if:

(Def. 3) For all x, y such that $x \in X$ and $y \in X$ holds $x \in y$ or $x = y$ or $y \in x$.

Let I_1 be a set. We say that I_1 is ordinal if and only if:

(Def. 4) I_1 is transitive and connected.

Let us mention that every set which is ordinal is also transitive and connected and every set which is transitive and connected is also ordinal.

We introduce number as a synonym of set.

Let us observe that there exists a number which is ordinal.

An ordinal number is an ordinal number.

In the sequel A, B, C denote ordinal numbers.

One can prove the following propositions:

(19)[4] For every transitive set A such that $A \in B$ and $B \in C$ holds $A \in C$.

(21)[5] For every transitive set x and for every ordinal number A such that $x \subset A$ holds $x \in A$.

(22) For every transitive set A and for all ordinal numbers B, C such that $A \subseteq B$ and $B \in C$ holds $A \in C$.

(23) If $a \in A$, then a is an ordinal number.

(24) $A \in B$ or $A = B$ or $B \in A$.

Let us consider A, B. Let us note that the predicate $A \subseteq B$ is connected.

The following three propositions are true:

(25) A and B are $\subseteq$-comparable.

(26) $A \subseteq B$ or $B \in A$.

(27) $\emptyset$ is an ordinal number.

One can verify that there exists an ordinal number which is empty.

One can check that every number which is empty is also ordinal.

One can verify that $\emptyset$ is ordinal.

We now state two propositions:

(29)[6] If x is an ordinal number, then $\mathrm{succ}\, x$ is an ordinal number.

(30) If x is ordinal, then $\bigcup x$ is ordinal.

Let us observe that there exists an ordinal number which is non empty.

Let us consider A. One can verify that $\mathrm{succ}\, A$ is non empty and ordinal and $\bigcup A$ is ordinal.

The following propositions are true:

(31) If for every x such that $x \in X$ holds x is an ordinal number and $x \subseteq X$, then X is ordinal.

(32) If $X \subseteq A$ and $X \neq \emptyset$, then there exists C such that $C \in X$ and for every B such that $B \in X$ holds $C \subseteq B$.

(33) $A \in B$ iff $\mathrm{succ}\, A \subseteq B$.

(34) $A \in \mathrm{succ}\, C$ iff $A \subseteq C$.

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[4] The propositions (15)–(18) have been removed.

[5] The proposition (20) has been removed.

[6] The proposition (28) has been removed.

In this article we present several logical schemes. The scheme *Ordinal Min* concerns a unary predicate $\mathcal{P}$, and states that:

There exists A such that $\mathcal{P}[A]$ and for every B such that $\mathcal{P}[B]$ holds $A \subseteq B$

provided the parameters meet the following requirement:

- There exists A such that $\mathcal{P}[A]$.

The scheme *Transfinite Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every A holds $\mathcal{P}[A]$

provided the parameters have the following property:

- For every A such that for every C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$.

The following propositions are true:

(35) For every X such that for every a such that $a \in X$ holds a is an ordinal number holds $\bigcup X$ is ordinal.

(36) For every X such that for every a such that $a \in X$ holds a is an ordinal number there exists A such that $X \subseteq A$.

(37) It is not true that there exists X such that for every x holds $x \in X$ iff x is an ordinal number.

(38) It is not true that there exists X such that for every A holds $A \in X$.

(39) For every X there exists A such that $A \notin X$ and for every B such that $B \notin X$ holds $A \subseteq B$.

Let A be a set. We say that A is limit if and only if:

(Def. 6)[7] $A = \bigcup A$.

We introduce A is a limit ordinal number as a synonym of A is limit.

Next we state two propositions:

(41)[8] For every A holds A is a limit ordinal number iff for every C such that $C \in A$ holds $\mathrm{succ}\, C \in A$.

(42) A is not a limit ordinal number iff there exists B such that $A = \mathrm{succ}\, B$.

In the sequel F denotes a function.

Let I_1 be a function. We say that I_1 is transfinite sequence-like if and only if:

(Def. 7) $\mathrm{dom}\, I_1$ is ordinal.

One can verify that there exists a function which is transfinite sequence-like.

A transfinite sequence is a transfinite sequence-like function.

Let us consider Z. A transfinite sequence is said to be a transfinite sequence of elements of Z if:

(Def. 8) $\mathrm{rng}\, \mathrm{it} \subseteq Z$.

Next we state the proposition

(45)[9] $\emptyset$ is a transfinite sequence of elements of Z.

In the sequel L, L_1 denote transfinite sequences.

Next we state the proposition

(46) If $\mathrm{dom}\, F$ is an ordinal number, then F is a transfinite sequence of elements of $\mathrm{rng}\, F$.

Let us consider L. Note that $\mathrm{dom}\, L$ is ordinal.

One can prove the following proposition

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[7] The definition (Def. 5) has been removed.

[8] The proposition (40) has been removed.

[9] The propositions (43) and (44) have been removed.

(47) If $X \subseteq Y$, then every transfinite sequence of elements of X is a transfinite sequence of elements of Y.

Let us consider L, A. Then $L \restriction A$ is a transfinite sequence of elements of $\operatorname{rng} L$.
The following proposition is true

(48) Let L be a transfinite sequence of elements of X and given A. Then $L \restriction A$ is a transfinite sequence of elements of X.

Let I_1 be a set. We say that I_1 is $\subseteq$-linear if and only if:

(Def. 9) For all sets x, y such that $x \in I_1$ and $y \in I_1$ holds x and y are $\subseteq$-comparable.

One can prove the following proposition

(49) Suppose for every a such that $a \in X$ holds a is a transfinite sequence and X is $\subseteq$-linear. Then $\bigcup X$ is a transfinite sequence.

Now we present three schemes. The scheme *TS Uniq* deals with an ordinal number $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and transfinite sequences $\mathcal{B}$, $\mathcal{C}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{B} = \mathcal{A}$ and for all B, L such that $B \in \mathcal{A}$ and $L = \mathcal{B} \restriction B$ holds $\mathcal{B}(B) = \mathcal{F}(L)$, and
- $\operatorname{dom} \mathcal{C} = \mathcal{A}$ and for all B, L such that $B \in \mathcal{A}$ and $L = \mathcal{C} \restriction B$ holds $\mathcal{C}(B) = \mathcal{F}(L)$.

The scheme *TS Exist* deals with an ordinal number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists L such that $\operatorname{dom} L = \mathcal{A}$ and for all B, L_1 such that $B \in \mathcal{A}$ and $L_1 = L \restriction B$ holds $L(B) = \mathcal{F}(L_1)$

for all values of the parameters.

The scheme *Func TS* deals with a transfinite sequence $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary functor $\mathcal{G}$ yielding a set, and states that:

For every B such that $B \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(B) = \mathcal{G}(\mathcal{A} \restriction B)$

provided the following conditions are met:

- For all A, a holds $a = \mathcal{F}(A)$ iff there exists L such that $a = \mathcal{G}(L)$ and $\operatorname{dom} L = A$ and for every B such that $B \in A$ holds $L(B) = \mathcal{G}(L \restriction B)$, and
- For every A such that $A \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

One can prove the following proposition

(50) $A \subset B$ or $A = B$ or $B \subset A$.

References

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