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On the Category of Posets

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Summary. In the paper the construction of a category of partially ordered sets is shown: in the second section according to [6] and in the third section according to the definition given in [17]. Some of useful notions such as monotone map and the set of monotone maps between relational structures are given.

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The articles [13], [8], [19], [20], [22], [4], [2], [14], [1], [7], [3], [11], [21], [12], [18], [6], [9], [15], [16], [5], [10], and [17] provide the notation and terminology for this paper.

1. Preliminaries

Let I_1 be a relational structure. We say that I_1 is discrete if and only if:

(Def. 1) The internal relation of $I_1 = \mathrm{id}_{\text{the carrier of } I_1}$.

Let us note that there exists a poset which is strict, discrete, and non empty and there exists a poset which is strict, discrete, and empty.

Observe that $\langle \emptyset, \mathrm{id}_\emptyset \rangle$ is empty. Let P be an empty relational structure. Note that the internal relation of P is empty.

Let us note that every relational structure which is empty is also discrete.

Let P be a relational structure and let I_1 be a subset of P. We say that I_1 is disconnected if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist subsets A, B of P such that

(i) $A \neq \emptyset$,

(ii) $B \neq \emptyset$,

(iii) $I_1 = A \cup B$,

(iv) A misses B, and

(v) the internal relation of $P =$ (the internal relation of P) $|^2 A \cup$ (the internal relation of P) $|^2 B$.

We introduce I_1 is connected as an antonym of I_1 is disconnected.

Let I_1 be a relational structure. We say that I_1 is disconnected if and only if:

(Def. 3) $\Omega_{(I_1)}$ is disconnected.

We introduce I_1 is connected as an antonym of I_1 is disconnected.

In the sequel T is a non empty relational structure and a is an element of T.

One can prove the following propositions:

(1) For every discrete non empty relational structure D_1 and for all elements x, y of D_1 holds $x \leq y$ iff $x = y$.

(2) For every binary relation R and for every set a such that R is an order in $\{a\}$ holds $R = \mathrm{id}_{\{a\}}$.

(3) If T is reflexive and $\Omega_T = \{a\}$, then T is discrete.

In the sequel a is a set.
Next we state two propositions:

(4) If $\Omega_T = \{a\}$, then T is connected.

(5) For every discrete non empty poset D_1 such that there exist elements a, b of D_1 such that $a \neq b$ holds D_1 is disconnected.

Let us observe that there exists a non empty poset which is strict and connected and there exists a non empty poset which is strict, disconnected, and discrete.

2. On the Category of Posets

Let I_1 be a set. We say that I_1 is poset-membered if and only if:

(Def. 4) For every set a such that $a \in I_1$ holds a is a non empty poset.

Let us observe that there exists a set which is non empty and poset-membered.
A set of posets is a poset-membered set.
Let P be a non empty set of posets. We see that the element of P is a non empty poset.
Let L_1, L_2 be relational structures and let f be a map from L_1 into L_2. We say that f is monotone if and only if:

(Def. 5) For all elements x, y of L_1 such that $x \leq y$ and for all elements a, b of L_2 such that $a = f(x)$ and $b = f(y)$ holds $a \leq b$.

In the sequel P denotes a non empty set of posets and A, B denote elements of P.
Let A, B be relational structures. The functor $B^A_{\leq}$ is defined by the condition (Def. 6).

(Def. 6) $a \in B^A_{\leq}$ if and only if there exists a map f from A into B such that $a = f$ and $f \in (\text{the carrier of } B)^{\text{the carrier of } A}$ and f is monotone.

We now state two propositions:

(6) For all non empty relational structures A, B, C and for all functions f, g such that $f \in B^A_{\leq}$ and $g \in C^B_{\leq}$ holds $g \cdot f \in C^A_{\leq}$.

(7) $\mathrm{id}_{\text{the carrier of } T} \in T^T_{\leq}$.

Let us consider T. Observe that $T^T_{\leq}$ is non empty.
Let X be a set. The functor $\mathrm{Carr}(X)$ yielding a set is defined by:

(Def. 7) $a \in \mathrm{Carr}(X)$ iff there exists a 1-sorted structure s such that $s \in X$ and $a =$ the carrier of s.

Let us consider P. Observe that $\mathrm{Carr}(P)$ is non empty.
Next we state four propositions:

(8) For every 1-sorted structure f holds $\mathrm{Carr}(\{f\}) = \{\text{the carrier of } f\}$.

(9) For all 1-sorted structures f, g holds $\mathrm{Carr}(\{f, g\}) = \{\text{the carrier of } f, \text{the carrier of } g\}$.

(10) $B^A_{\leq} \subseteq \mathrm{Funcs}\,\mathrm{Carr}(P)$.

(11) For all relational structures A, B holds $B^A_{\leq} \subseteq (\text{the carrier of } B)^{\text{the carrier of } A}$.

Let A, B be non empty posets. Note that $B^{A}_{\leq}$ is functional.

Let P be a non empty set of posets. The functor $\mathrm{POSCat}(P)$ yields a strict category with triple-like morphisms and is defined by the conditions (Def. 8).

(Def. 8)(i) The objects of $\mathrm{POSCat}(P) = P$,

(ii) for all elements a, b of P and for every element f of $\mathrm{FuncsCarr}(P)$ such that $f \in b^{a}_{\leq}$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of $\mathrm{POSCat}(P)$,

(iii) for every morphism m of $\mathrm{POSCat}(P)$ there exist elements a, b of P and there exists an element f of $\mathrm{FuncsCarr}(P)$ such that $m = \langle\langle a, b\rangle, f\rangle$ and $f \in b^{a}_{\leq}$, and

(iv) for all morphisms m_1, m_2 of $\mathrm{POSCat}(P)$ and for all elements a_1, a_2, a_3 of P and for all elements f_1, f_2 of $\mathrm{FuncsCarr}(P)$ such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, f_2 \cdot f_1\rangle$.

3. On the Alternative Category of Posets

In this article we present several logical schemes. The scheme *AltCatEx* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a functional set, and states that:

There exists a strict category structure C such that

(i) the carrier of $C = \mathcal{A}$, and

(ii) for all elements i, j of $\mathcal{A}$ holds (the arrows of C)$(i, j) = \mathcal{F}(i, j)$ and for all elements i, j, k of $\mathcal{A}$ holds (the composition of C)$(i, j, k) = \mathrm{FuncComp}(\mathcal{F}(i, j), \mathcal{F}(j, k))$

provided the parameters meet the following requirement:

- For all elements i, j, k of $\mathcal{A}$ and for all functions f, g such that $f \in \mathcal{F}(i, j)$ and $g \in \mathcal{F}(j, k)$ holds $g \cdot f \in \mathcal{F}(i, k)$.

The scheme *AltCatUniq* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a functional set, and states that:

Let C_1, C_2 be strict category structures. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,

(ii) for all elements i, j of $\mathcal{A}$ holds (the arrows of C_1)$(i, j) = \mathcal{F}(i, j)$ and for all elements i, j, k of $\mathcal{A}$ holds (the composition of C_1)$(i, j, k) = \mathrm{FuncComp}(\mathcal{F}(i, j), \mathcal{F}(j, k))$,

(iii) the carrier of $C_2 = \mathcal{A}$, and

(iv) for all elements i, j of $\mathcal{A}$ holds (the arrows of C_2)$(i, j) = \mathcal{F}(i, j)$ and for all elements i, j, k of $\mathcal{A}$ holds (the composition of C_2)$(i, j, k) = \mathrm{FuncComp}(\mathcal{F}(i, j), \mathcal{F}(j, k))$.

Then $C_1 = C_2$

for all values of the parameters.

Let P be a non empty set of posets. The functor $\mathrm{POSAltCat}(P)$ yielding a strict category structure is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of $\mathrm{POSAltCat}(P) = P$, and

(ii) for all elements i, j of P holds (the arrows of $\mathrm{POSAltCat}(P)$)$(i, j) = j^{i}_{\leq}$ and for all elements i, j, k of P holds (the composition of $\mathrm{POSAltCat}(P)$)$(i, j, k) = \mathrm{FuncComp}(j^{i}_{\leq}, k^{j}_{\leq})$.

Let P be a non empty set of posets. Observe that $\mathrm{POSAltCat}(P)$ is transitive and non empty.

Let P be a non empty set of posets. Note that $\mathrm{POSAltCat}(P)$ is associative and has units.

We now state the proposition

(12) Let o_1, o_2 be objects of $\mathrm{POSAltCat}(P)$ and A, B be elements of P. If $o_1 = A$ and $o_2 = B$, then $\langle o_1, o_2\rangle \subseteq (\text{the carrier of } B)^{\text{the carrier of } A}$.

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