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Basic Notions and Properties of Orthoposets[1]

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Summary. Orthoposets are defined. The approach is the standard one via order relation similar to common text books on algebra like [9].

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The articles [12], [14], [6], [3], [15], [17], [4], [16], [5], [13], [10], [8], [2], [7], [1], and [11] provide the notation and terminology for this paper.

1. General Notions and Properties

In this paper S, X denote non empty sets and R denotes a binary relation on X.

We introduce orthorelational structures which are extensions of relational structure and ComplStr and are systems

$\langle$ a carrier, an internal relation, a complement operation $\rangle$,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the complement operation is a unary operation on the carrier.

Let A, B be sets. The functor $\emptyset_{A,B}$ yielding a relation between A and B is defined by:

(Def. 1) $\emptyset_{A,B} = \emptyset$.

The functor $\Omega_B(A)$ yielding a relation between A and B is defined as follows:

(Def. 2) $\Omega_B(A) = [:A, B:]$.

We now state a number of propositions:

(1) $\mathrm{field}(\mathrm{id}_X) = X$.

(2) $\mathrm{id}_{\{\emptyset\}} = \{\langle\emptyset, \emptyset\rangle\}$.

(3) $\mathrm{op}_1 = \{\langle\emptyset, \emptyset\rangle\}$.

(4) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of L. If $x \leq y$, then $\sup\{x, y\} = y$ and $\inf\{x, y\} = x$.

(5) For every binary relation R holds $\mathrm{dom}\, R \subseteq \mathrm{field}\, R$ and $\mathrm{rng}\, R \subseteq \mathrm{field}\, R$.

(6) For all sets A, B holds $\mathrm{field}(\emptyset_{A,B}) = \emptyset$.

(7) If R is reflexive in X, then R is reflexive and $\mathrm{field}\, R = X$.

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(8) If R is symmetric in X, then R is symmetric.

(9) If R is symmetric and $\mathrm{field} R \subseteq S$, then R is symmetric in S.

(10) If R is antisymmetric and $\mathrm{field} R \subseteq S$, then R is antisymmetric in S.

(11) If R is antisymmetric in X, then R is antisymmetric.

(12) If R is transitive and $\mathrm{field} R \subseteq S$, then R is transitive in S.

(13) If R is transitive in X, then R is transitive.

(14) If R is asymmetric and $\mathrm{field} R \subseteq S$, then R is asymmetric in S.

(15) If R is asymmetric in X, then R is asymmetric.

(16) If R is irreflexive and $\mathrm{field} R \subseteq S$, then R is irreflexive in S.

(17) If R is irreflexive in X, then R is irreflexive.

Let X be a set. Observe that there exists a binary relation on X which is irreflexive, asymmetric, and transitive.

Let us consider X, R and let C be a unary operation on X. One can verify that $\langle X, R, C \rangle$ is non empty.

Let us observe that there exists a orthorelational structure which is non empty and strict.

Let us consider X and let f be a function from X into X. We say that f is dneg if and only if:

(Def. 3) For every element x of X holds $f(f(x)) = x$.

We introduce f is involutive as a synonym of f is dneg.

Next we state two propositions:

(19)[1] op_1 is dneg.

(20) id_X is dneg.

Let O be a non empty orthorelational structure. One can check that there exists a map from O into O which is dneg.

The strict orthorelational structure TrivOrthoRelStr is defined by:

(Def. 5)[2] $\mathrm{TrivOrthoRelStr} = \langle \{\emptyset\}, \mathrm{id}_{\{\emptyset\}}, \mathrm{op}_1 \rangle$.

We introduce TrivPoset as a synonym of TrivOrthoRelStr.

Let us observe that TrivOrthoRelStr is non empty and trivial.

The strict orthorelational structure TrivAsymOrthoRelStr is defined by:

(Def. 6) $\mathrm{TrivAsymOrthoRelStr} = \langle \{\emptyset\}, \emptyset_{\{\emptyset\},\{\emptyset\}}, \mathrm{op}_1 \rangle$.

One can verify that TrivAsymOrthoRelStr is non empty.

Let O be a non empty orthorelational structure. We say that O is Dneg if and only if:

(Def. 7) There exists a map f from O into O such that $f =$ the complement operation of O and f is dneg.

One can prove the following proposition

(21) TrivOrthoRelStr is Dneg.

Let us mention that TrivOrthoRelStr is Dneg.

Let us observe that there exists a non empty orthorelational structure which is Dneg.

Let O be a non empty relational structure. We say that O is SubReFlexive if and only if:

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[1] The proposition (18) has been removed.

[2] The definition (Def. 4) has been removed.

(Def. 10)[3] The internal relation of O is reflexive.

In the sequel O denotes a non empty relational structure.
Next we state two propositions:

(22) If O is reflexive, then O is SubReFlexive.

(23) TrivOrthoRelStr is reflexive.

Let us note that TrivOrthoRelStr is reflexive.
Let us mention that there exists a non empty orthorelational structure which is reflexive and strict.
Let us consider O. We say that O is SubIrreFlexive if and only if:

(Def. 12)[4] The internal relation of O is irreflexive.

Let us observe that O is irreflexive if and only if:

(Def. 13) The internal relation of O is irreflexive in the carrier of O.

We now state two propositions:

(24) If O is irreflexive, then O is SubIrreFlexive.

(25) TrivAsymOrthoRelStr is irreflexive.

Let us observe that every non empty orthorelational structure which is irreflexive is also SubIrreFlexive.
Let us mention that TrivAsymOrthoRelStr is irreflexive.
One can check that there exists a non empty orthorelational structure which is irreflexive and strict.
Let O be a non empty relational structure. We say that O is SubSymmetric if and only if:

(Def. 14) The internal relation of O is a symmetric binary relation on the carrier of O.

Next we state two propositions:

(26) If O is symmetric, then O is SubSymmetric.

(27) TrivOrthoRelStr is symmetric.

One can verify that every non empty orthorelational structure which is symmetric is also SubSymmetric.
Let us observe that there exists a non empty orthorelational structure which is symmetric and strict.
Let us consider O. We say that O is SubAntisymmetric if and only if:

(Def. 16)[5] The internal relation of O is an antisymmetric binary relation on the carrier of O.

We now state two propositions:

(28) If O is antisymmetric, then O is SubAntisymmetric.

(29) TrivOrthoRelStr is antisymmetric.

Let us mention that every non empty orthorelational structure which is antisymmetric is also SubAntisymmetric.
Let us observe that TrivOrthoRelStr is symmetric.
Let us note that there exists a non empty orthorelational structure which is symmetric, antisymmetric, and strict.
Let us consider O. We say that O is Asymmetric if and only if:

[3] The definitions (Def. 8) and (Def. 9) have been removed.
[4] The definition (Def. 11) has been removed.
[5] The definition (Def. 15) has been removed.

(Def. 19)[6] The internal relation of O is asymmetric in the carrier of O.

One can prove the following two propositions:

(30) If O is Asymmetric, then O is asymmetric.

(31) TrivAsymOrthoRelStr is Asymmetric.

Let us observe that every non empty orthorelational structure which is Asymmetric is also asymmetric.

Let us mention that TrivAsymOrthoRelStr is Asymmetric.

Let us note that there exists a non empty orthorelational structure which is Asymmetric and strict.

Let us consider O. We say that O is SubTransitive if and only if:

(Def. 20) The internal relation of O is a transitive binary relation on the carrier of O.

Next we state two propositions:

(32) If O is transitive, then O is SubTransitive.

(33) TrivOrthoRelStr is transitive.

Let us note that every non empty orthorelational structure which is transitive is also SubTransitive.

Let us observe that there exists a non empty orthorelational structure which is reflexive, symmetric, antisymmetric, transitive, and strict.

One can prove the following proposition

(34) TrivAsymOrthoRelStr is transitive.

One can check that TrivAsymOrthoRelStr is irreflexive, Asymmetric, and transitive.

One can verify that there exists a non empty orthorelational structure which is irreflexive, Asymmetric, transitive, and strict.

The following four propositions are true:

(35) If O is SubSymmetric and SubTransitive, then O is SubReFlexive.

(36) If O is SubIrreFlexive and SubTransitive, then O is asymmetric.

(37) If O is asymmetric, then O is SubIrreFlexive.

(38) If O is reflexive and SubSymmetric, then O is symmetric.

One can check that every non empty orthorelational structure which is reflexive and SubSymmetric is also symmetric.

The following proposition is true

(39) If O is reflexive and SubAntisymmetric, then O is antisymmetric.

Let us observe that every non empty orthorelational structure which is reflexive and SubAntisymmetric is also antisymmetric.

The following proposition is true

(40) If O is reflexive and SubTransitive, then O is transitive.

One can verify that every non empty orthorelational structure which is reflexive and SubTransitive is also transitive.

Next we state the proposition

[6] The definitions (Def. 17) and (Def. 18) have been removed.

(41) If O is irreflexive and SubTransitive, then O is transitive.

Let us mention that every non empty orthorelational structure which is irreflexive and SubTransitive is also transitive.

One can prove the following proposition

(42) If O is irreflexive and asymmetric, then O is Asymmetric.

Let us note that every non empty orthorelational structure which is irreflexive and asymmetric is also Asymmetric.

2. Basic Poset Notions

Let us consider O. We say that O is SubQuasiOrdered if and only if:

(Def. 22)[7] O is SubReFlexive and SubTransitive.

We introduce O is SubQuasiordered, O is SubPreOrdered, O is SubPreordered, and O is Subpreordered as synonyms of O is SubQuasiOrdered.

Let us consider O. We say that O is QuasiOrdered if and only if:

(Def. 23) O is reflexive and transitive.

We introduce O is Quasiordered, O is PreOrdered, and O is Preordered as synonyms of O is QuasiOrdered.

One can prove the following proposition

(43) If O is QuasiOrdered, then O is SubQuasiOrdered.

One can check that every non empty orthorelational structure which is QuasiOrdered is also SubQuasiOrdered.

Let us observe that TrivOrthoRelStr is QuasiOrdered.

In the sequel O is a non empty orthorelational structure.

Let us consider O. We say that O is QuasiPure if and only if:

(Def. 24) O is Dneg and QuasiOrdered.

Let us mention that there exists a non empty orthorelational structure which is QuasiPure, Dneg, QuasiOrdered, and strict.

One can verify that TrivOrthoRelStr is QuasiPure.

A QuasiPureOrthoRelStr is a QuasiPure non empty orthorelational structure.

Let us consider O. We say that O is SubPartialOrdered if and only if:

(Def. 25) O is reflexive, SubAntisymmetric, and SubTransitive.

We introduce O is SubPartialordered as a synonym of O is SubPartialOrdered.

Let us consider O. We say that O is PartialOrdered if and only if:

(Def. 26) O is reflexive, antisymmetric, and transitive.

We introduce O is Partialordered as a synonym of O is PartialOrdered.

The following proposition is true

(44) O is SubPartialOrdered iff O is PartialOrdered.

One can check that every non empty orthorelational structure which is SubPartialOrdered is also PartialOrdered and every non empty orthorelational structure which is PartialOrdered is also SubPartialOrdered.

Let us note that every non empty orthorelational structure which is PartialOrdered is also reflexive, antisymmetric, and transitive and every non empty orthorelational structure which is reflexive, antisymmetric, and transitive is also PartialOrdered.

Let us consider O. We say that O is Pure if and only if:

[7] The definition (Def. 21) has been removed.

(Def. 27) O is Dneg and PartialOrdered.

Let us note that there exists a non empty orthorelational structure which is Pure, Dneg, PartialOrdered, and strict.

Let us mention that TrivOrthoRelStr is Pure.

A PureOrthoRelStr is a Pure non empty orthorelational structure.

Let us consider O. We say that O is SubStrictPartialOrdered if and only if:

(Def. 28) O is asymmetric and SubTransitive.

Let us consider O. We say that O is StrictPartialOrdered if and only if:

(Def. 29) O is Asymmetric and transitive.

We introduce O is Strictpartialordered, O is StrictOrdered, and O is Strictordered as synonyms of O is StrictPartialOrdered.

The following proposition is true

(45) If O is StrictPartialOrdered, then O is SubStrictPartialOrdered.

Let us observe that every non empty orthorelational structure which is StrictPartialOrdered is also SubStrictPartialOrdered.

Next we state the proposition

(46) If O is SubStrictPartialOrdered, then O is SubIrreFlexive.

Let us observe that every non empty orthorelational structure which is SubStrictPartialOrdered is also SubIrreFlexive.

We now state the proposition

(47) If O is irreflexive and SubStrictPartialOrdered, then O is StrictPartialOrdered.

One can verify that every non empty orthorelational structure which is irreflexive and SubStrictPartialOrdered is also StrictPartialOrdered.

We now state the proposition

(48) If O is StrictPartialOrdered, then O is irreflexive.

Let us mention that every non empty orthorelational structure which is StrictPartialOrdered is also irreflexive.

Let us observe that TrivAsymOrthoRelStr is irreflexive and StrictPartialOrdered.

One can check that there exists a non empty strict orthorelational structure which is irreflexive and StrictPartialOrdered.

In the sequel P_1 is a PartialOrdered non empty orthorelational structure and Q_1 is a QuasiOrdered non empty orthorelational structure.

The following two propositions are true:

(49) If Q_1 is SubAntisymmetric, then Q_1 is PartialOrdered.

(50) P_1 is a poset.

Let us observe that every non empty orthorelational structure which is PartialOrdered is also reflexive, transitive, and antisymmetric.

Let P_1 be a PartialOrdered non empty orthorelational structure and let f be a unary operation on the carrier of P_1. We say that f is Orderinvolutive if and only if:

(Def. 33)[8] f is a dneg map from P_1 into P_1 and an antitone map from P_1 into P_1.

Let us consider P_1. We say that P_1 is OrderInvolutive if and only if:

[8] The definitions (Def. 30)–(Def. 32) have been removed.

(Def. 34) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is Orderinvolutive.

We now state the proposition

(51) The complement operation of TrivOrthoRelStr is Orderinvolutive.

Let us note that TrivOrthoRelStr is OrderInvolutive.

Let us observe that there exists a PartialOrdered non empty orthorelational structure which is OrderInvolutive and Pure.

A PreOrthoPoset is an OrderInvolutive Pure PartialOrdered non empty orthorelational structure.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is QuasiOrthoComplement on P_1 if and only if:

(Def. 35) f is Orderinvolutive and for every element y of P_1 holds sup $\{y, f(y)\}$ exists in P_1 and inf $\{y, f(y)\}$ exists in P_1.

Let us consider P_1. We say that P_1 is QuasiOrthocomplemented if and only if:

(Def. 36) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is QuasiOrthoComplement on P_1.

The following proposition is true

(52) TrivOrthoRelStr is QuasiOrthocomplemented.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is OrthoComplement on P_1 if and only if the conditions (Def. 37) are satisfied.

(Def. 37)(i) f is Orderinvolutive, and

(ii) for every element y of P_1 holds sup $\{y, f(y)\}$ exists in P_1 and inf $\{y, f(y)\}$ exists in P_1 and $\bigsqcup_{P_1}\{y, f(y)\}$ is a maximum of the carrier of P_1 and $\bigsqcap_{P_1}\{y, f(y)\}$ is a minimum of the carrier of P_1.

We introduce f is OCompl on P_1 as a synonym of f is OrthoComplement on P_1.

Let us consider P_1. We say that P_1 is Orthocomplemented if and only if:

(Def. 38) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is OrthoComplement on P_1.

We introduce P_1 is Ocompl as a synonym of P_1 is Orthocomplemented.

The following propositions are true:

(53) Let f be a unary operation on the carrier of P_1. If f is OrthoComplement on P_1, then f is QuasiOrthoComplement on P_1.

(54) TrivOrthoRelStr is Orthocomplemented.

Let us note that TrivOrthoRelStr is QuasiOrthocomplemented and Orthocomplemented.

Let us observe that there exists a PartialOrdered non empty orthorelational structure which is Orthocomplemented and QuasiOrthocomplemented.

A QuasiOrthoPoset is a QuasiOrthocomplemented PartialOrdered non empty orthorelational structure. An orthoposet is an Orthocomplemented PartialOrdered non empty orthorelational structure.

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