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Algebra of Normal Forms

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Summary. We mean by a normal form a finite set of ordered pairs of subsets of a fixed set that fulfils two conditions: elements of it consist of disjoint sets and element of it are incomparable w.r.t. inclusion. The underlying set corresponds to a set of propositional variables but it is arbitrary. The correspondents to a normal form of a formula, e.g. a disjunctive normal form is as follows. The normal form is the set of disjuncts and a disjunct is an ordered pair consisting of the sets of propositional variables that occur in the disjunct non-negated and negated. The requirement that the element of a normal form consists of disjoint sets means that contradictory disjuncts have been removed and the second condition means that the absorption law has been used to shorten the normal form. We construct a lattice $\langle \mathbb{N}, \sqcup, \sqcap \rangle$, where $a \sqcup b = \mu(a \cup b)$ and $a \sqcap b = \mu c$, c being set of all pairs $\langle X_1 \cup Y_1, X_2 \cup Y_2 \rangle$, $\langle X_1, X_2 \rangle \in a$ and $\langle Y_1, Y_2 \rangle \in b$, which consist of disjoiny sets. μa denotes here the set of all minimal, w.r.t. inclusion, elements of a. We prove that the lattice of normal forms over a set defined in this way is distributive and that $\emptyset$ is the minimal element of it.

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The articles [6], [3], [9], [7], [10], [2], [1], [4], [8], [5], and [11] provide the notation and terminology for this paper.

In this paper A, B denote non empty preboolean sets and x, y denote elements of $[:A, B:]$.

Let us consider A, B, x, y. The predicate $x \subseteq y$ is defined as follows:

(Def. 1) $x_{\mathbf{1}} \subseteq y_{\mathbf{1}}$ and $x_{\mathbf{2}} \subseteq y_{\mathbf{2}}$.

Let us note that the predicate $x \subseteq y$ is reflexive. The functor $x \cup y$ yields an element of $[:A, B:]$ and is defined as follows:

(Def. 2) $x \cup y = \langle x_{\mathbf{1}} \cup y_{\mathbf{1}}, x_{\mathbf{2}} \cup y_{\mathbf{2}} \rangle$.

Let us notice that the functor $x \cup y$ is commutative and idempotent. The functor $x \cap y$ yielding an element of $[:A, B:]$ is defined as follows:

(Def. 3) $x \cap y = \langle x_{\mathbf{1}} \cap y_{\mathbf{1}}, x_{\mathbf{2}} \cap y_{\mathbf{2}} \rangle$.

Let us observe that the functor $x \cap y$ is commutative and idempotent. The functor $x \setminus y$ yielding an element of $[:A, B:]$ is defined by:

(Def. 4) $x \setminus y = \langle x_{\mathbf{1}} \setminus y_{\mathbf{1}}, x_{\mathbf{2}} \setminus y_{\mathbf{2}} \rangle$.

The functor $x \dot{-} y$ yielding an element of $[:A, B:]$ is defined by:

(Def. 5) $x \dot{-} y = \langle x_{\mathbf{1}} \dot{-} y_{\mathbf{1}}, x_{\mathbf{2}} \dot{-} y_{\mathbf{2}} \rangle$.

Let us note that the functor $x \dot{-} y$ is commutative.

In the sequel X is a set and a, b, c are elements of $[: A, B :]$.

Next we state a number of propositions:

(4)[1] If $a \subseteq b$ and $b \subseteq a$, then $a = b$.

(5) If $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$.

(10)[2] $(a \cup b)_{\mathbf{1}} = a_{\mathbf{1}} \cup b_{\mathbf{1}}$ and $(a \cup b)_{\mathbf{2}} = a_{\mathbf{2}} \cup b_{\mathbf{2}}$.

(11) $(a \cap b)_{\mathbf{1}} = a_{\mathbf{1}} \cap b_{\mathbf{1}}$ and $(a \cap b)_{\mathbf{2}} = a_{\mathbf{2}} \cap b_{\mathbf{2}}$.

(12) $(a \setminus b)_{\mathbf{1}} = a_{\mathbf{1}} \setminus b_{\mathbf{1}}$ and $(a \setminus b)_{\mathbf{2}} = a_{\mathbf{2}} \setminus b_{\mathbf{2}}$.

(13) $(a \dot{-} b)_{\mathbf{1}} = a_{\mathbf{1}} \dot{-} b_{\mathbf{1}}$ and $(a \dot{-} b)_{\mathbf{2}} = a_{\mathbf{2}} \dot{-} b_{\mathbf{2}}$.

(16)[3] $(a \cup b) \cup c = a \cup (b \cup c)$.

(19)[4] $(a \cap b) \cap c = a \cap (b \cap c)$.

(20) $a \cap (b \cup c) = a \cap b \cup a \cap c$.

(21) $a \cup b \cap a = a$.

(22) $a \cap (b \cup a) = a$.

(24)[5] $a \cup b \cap c = (a \cup b) \cap (a \cup c)$.

(25) If $a \subseteq c$ and $b \subseteq c$, then $a \cup b \subseteq c$.

(26) $a \subseteq a \cup b$ and $b \subseteq a \cup b$.

(27) If $a = a \cup b$, then $b \subseteq a$.

(28) If $a \subseteq b$, then $c \cup a \subseteq c \cup b$ and $a \cup c \subseteq b \cup c$.

(29) $(a \setminus b) \cup b = a \cup b$.

(30) If $a \setminus b \subseteq c$, then $a \subseteq b \cup c$.

(31) If $a \subseteq b \cup c$, then $a \setminus c \subseteq b$.

In the sequel a denotes an element of $[: \mathrm{Fin}\, X, \mathrm{Fin}\, X :]$.

Let A be a set. The functor $\mathrm{FinUnion}_A$ yielding a binary operation on $[: \mathrm{Fin}\, A, \mathrm{Fin}\, A :]$ is defined by:

(Def. 6) For all elements x, y of $[: \mathrm{Fin}\, A, \mathrm{Fin}\, A :]$ holds $\mathrm{FinUnion}_A(x, y) = x \cup y$.

In the sequel A denotes a set.

Let X be a non empty set, let A be a set, let B be an element of $\mathrm{Fin}\, X$, and let f be a function from X into $[: \mathrm{Fin}\, A, \mathrm{Fin}\, A :]$. The functor $\mathrm{FinUnion}(B, f)$ yields an element of $[: \mathrm{Fin}\, A, \mathrm{Fin}\, A :]$ and is defined by:

(Def. 7) $\mathrm{FinUnion}(B, f) = \mathrm{FinUnion}_A \text{-} \sum_B f$.

Let A be a set. One can check that $\mathrm{FinUnion}_A$ is commutative, idempotent, and associative.

We now state several propositions:

[1] The propositions (1)–(3) have been removed.
[2] The propositions (6)–(9) have been removed.
[3] The propositions (14) and (15) have been removed.
[4] The propositions (17) and (18) have been removed.
[5] The proposition (23) has been removed.

(35)[6] Let X be a non empty set, f be a function from X into $[: \operatorname{Fin} A, \operatorname{Fin} A :]$, B be an element of $\operatorname{Fin} X$, and x be an element of X. If $x \in B$, then $f(x) \subseteq \operatorname{FinUnion}(B, f)$.

(36) $\langle \emptyset_A, \emptyset_A \rangle$ is a unity w.r.t. $\operatorname{FinUnion}_A$.

(37) $\operatorname{FinUnion}_A$ has a unity.

(38) $\mathbf{1}_{\operatorname{FinUnion}_A} = \langle \emptyset_A, \emptyset_A \rangle$.

(39) For every element x of $[: \operatorname{Fin} A, \operatorname{Fin} A :]$ holds $\mathbf{1}_{\operatorname{FinUnion}_A} \subseteq x$.

(40) Let X be a non empty set, f be a function from X into $[: \operatorname{Fin} A, \operatorname{Fin} A :]$, B be an element of $\operatorname{Fin} X$, and c be an element of $[: \operatorname{Fin} A, \operatorname{Fin} A :]$. If for every element x of X such that $x \in B$ holds $f(x) \subseteq c$, then $\operatorname{FinUnion}(B, f) \subseteq c$.

(41) Let X be a non empty set, B be an element of $\operatorname{Fin} X$, and f, g be functions from X into $[: \operatorname{Fin} A, \operatorname{Fin} A :]$. If $f \restriction B = g \restriction B$, then $\operatorname{FinUnion}(B, f) = \operatorname{FinUnion}(B, g)$.

Let us consider X. The functor $\operatorname{DP}(X)$ yields a subset of $[: \operatorname{Fin} X, \operatorname{Fin} X :]$ and is defined as follows:

(Def. 8) $\operatorname{DP}(X) = \{a : a_{\mathbf{1}} \text{ misses } a_{\mathbf{2}}\}$.

Let us consider X. One can check that $\operatorname{DP}(X)$ is non empty.
Next we state the proposition

(42) For every element y of $[: \operatorname{Fin} X, \operatorname{Fin} X :]$ holds $y \in \operatorname{DP}(X)$ iff $y_{\mathbf{1}}$ misses $y_{\mathbf{2}}$.

In the sequel x, y are elements of $[: \operatorname{Fin} X, \operatorname{Fin} X :]$ and a, b are elements of $\operatorname{DP}(X)$.
Next we state several propositions:

(43) If $y \in \operatorname{DP}(X)$ and $x \in \operatorname{DP}(X)$, then $y \cup x \in \operatorname{DP}(X)$ iff $y_{\mathbf{1}} \cap x_{\mathbf{2}} \cup x_{\mathbf{1}} \cap y_{\mathbf{2}} = \emptyset$.

(44) $a_{\mathbf{1}}$ misses $a_{\mathbf{2}}$.

(45) If $x \subseteq b$, then x is an element of $\operatorname{DP}(X)$.

(46) It is not true that there exists a set x such that $x \in a_{\mathbf{1}}$ and $x \in a_{\mathbf{2}}$.

(47) If $a \cup b \notin \operatorname{DP}(X)$, then there exists an element p of X such that $p \in a_{\mathbf{1}}$ and $p \in b_{\mathbf{2}}$ or $p \in b_{\mathbf{1}}$ and $p \in a_{\mathbf{2}}$.

(49)[7] If $x_{\mathbf{1}}$ misses $x_{\mathbf{2}}$, then x is an element of $\operatorname{DP}(X)$.

(50) For all sets V, W such that $V \subseteq a_{\mathbf{1}}$ and $W \subseteq a_{\mathbf{2}}$ holds $\langle V, W \rangle$ is an element of $\operatorname{DP}(X)$.

For simplicity, we adopt the following rules: A denotes a set, x denotes an element of $[: \operatorname{Fin} A, \operatorname{Fin} A :]$, a, b, c, s, t denote elements of $\operatorname{DP}(A)$, and B, C, D denote elements of $\operatorname{Fin} \operatorname{DP}(A)$.
Let us consider A. The normal forms over A yielding a subset of $\operatorname{Fin} \operatorname{DP}(A)$ is defined by:

(Def. 9) The normal forms over $A = \{B : a \in B \wedge b \in B \wedge a \subseteq b \Rightarrow a = b\}$.

Let us consider A. Observe that the normal forms over A is non empty.
In the sequel K, L, M denote elements of the normal forms over A.
The following three propositions are true:

(51) $\emptyset \in$ the normal forms over A.

(52) If $B \in$ the normal forms over A and $a \in B$ and $b \in B$ and $a \subseteq b$, then $a = b$.

[6] The propositions (32)–(34) have been removed.
[7] The proposition (48) has been removed.

(53) If for all a, b such that $a \in B$ and $b \in B$ and $a \subseteq b$ holds $a = b$, then $B \in$ the normal forms over A.

Let us consider A, B. The functor μB yielding an element of the normal forms over A is defined by:

(Def. 10) $\mu B = \{t : s \in B \wedge s \subseteq t \Leftrightarrow s = t\}$.

Let us consider C. The functor $B \frown C$ yielding an element of $\mathrm{Fin}\,\mathrm{DP}(A)$ is defined as follows:

(Def. 11) $B \frown C = \mathrm{DP}(A) \cap \{s \cup t : s \in B \wedge t \in C\}$.

One can prove the following propositions:

(55)[8] If $x \in B \frown C$, then there exist b, c such that $b \in B$ and $c \in C$ and $x = b \cup c$.

(56) If $b \in B$ and $c \in C$ and $b \cup c \in \mathrm{DP}(A)$, then $b \cup c \in B \frown C$.

(58)[9] If $a \in \mu B$, then $a \in B$ and if $b \in B$ and $b \subseteq a$, then $b = a$.

(59) If $a \in \mu B$, then $a \in B$.

(60) If $a \in \mu B$ and $b \in B$ and $b \subseteq a$, then $b = a$.

(61) If $a \in B$ and for every b such that $b \in B$ and $b \subseteq a$ holds $b = a$, then $a \in \mu B$.

Let A be a non empty set, let B be a non empty subset of A, let O be a binary operation on B, and let a, b be elements of B. Then $O(a, b)$ is an element of B.

The following propositions are true:

(64)[10] $\mu B \subseteq B$.

(65) If $b \in B$, then there exists c such that $c \subseteq b$ and $c \in \mu B$.

(66) $\mu K = K$.

(67) $\mu(B \cup C) \subseteq \mu B \cup C$.

(68) $\mu(\mu B \cup C) = \mu(B \cup C)$.

(69) $\mu(B \cup \mu C) = \mu(B \cup C)$.

(70) If $B \subseteq C$, then $B \frown D \subseteq C \frown D$.

(71) $\mu(B \frown C) \subseteq \mu B \frown C$.

(72) $B \frown C = C \frown B$.

(73) If $B \subseteq C$, then $D \frown B \subseteq D \frown C$.

(74) $\mu(\mu B \frown C) = \mu(B \frown C)$.

(75) $\mu(B \frown \mu C) = \mu(B \frown C)$.

(76) $K \frown (L \frown M) = (K \frown L) \frown M$.

(77) $K \frown (L \cup M) = K \frown L \cup K \frown M$.

(78) $B \subseteq B \frown B$.

(79) $\mu(K \frown K) = \mu K$.

[8] The proposition (54) has been removed.
[9] The proposition (57) has been removed.
[10] The propositions (62) and (63) have been removed.

Let us consider A. The lattice of normal forms over A yields a strict lattice structure and is defined by the conditions (Def. 14).

(Def. 14)[11](i) The carrier of the lattice of normal forms over $A =$ the normal forms over A, and

(ii) for all elements B, C of the normal forms over A holds (the join operation of the lattice of normal forms over $A)(B, C) = \mu(B \cup C)$ and (the meet operation of the lattice of normal forms over $A)(B, C) = \mu(B \frown C)$.

Let us consider A. Note that the lattice of normal forms over A is non empty.

Let us consider A. Observe that the lattice of normal forms over A is lattice-like.

Let us consider A. One can verify that the lattice of normal forms over A is distributive and lower-bounded.

One can prove the following two propositions:

(85)[12] $\emptyset$ is an element of the lattice of normal forms over A.

(86) $\perp_{\text{the lattice of normal forms over } A} = \emptyset$.

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[11] The definitions (Def. 12) and (Def. 13) have been removed.

[12] The propositions (80)–(84) have been removed.