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# The Class of Series – Parallel Graphs. Part I

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**Summary.** The paper introduces some preliminary notions concerning the graph theory according to [18]. We define Necklace $n$ as a graph with vertex $\{1,2,3,\ldots,n\}$ and edge set $\{(1,2),(2,3),\ldots,(n-1,n)\}$. The goal of the article is to prove that Necklace $n$ and Complement of Necklace $n$ are isomorphic for $n=0,1,4$.

MML Identifier: `NECKLACE`.

WWW: `http://mizar.org/JFM/Vol14/necklace.html`

The articles [21], [20], [24], [11], [1], [15], [6], [3], [22], [2], [23], [25], [27], [17], [7], [12], [19], [26], [8], [10], [9], [13], [4], [5], [14], and [16] provide the notation and terminology for this paper.

## 1. Preliminaries

We follow the rules: $n$ denotes a natural number and $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $y_1$, $y_2$, $y_3$ denote sets.

Let $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ be sets. We say that $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ are mutually different if and only if:

(Def. 1) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_4 \neq x_5$.

We now state several propositions:

(1) If $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ are mutually different, then $\mathrm{card}\{x_1,x_2,x_3,x_4,x_5\} = 5$.

(2) $4 = \{0,1,2,3\}$.

(3) $[:\{x_1,x_2,x_3\}, \{y_1,y_2,y_3\}:] = \{\langle x_1, y_1\rangle, \langle x_1, y_2\rangle, \langle x_1, y_3\rangle, \langle x_2, y_1\rangle, \langle x_2, y_2\rangle, \langle x_2, y_3\rangle, \langle x_3, y_1\rangle, \langle x_3, y_2\rangle, \langle x_3, y_3\rangle\}$.

(4) For every set $x$ and for every natural number $n$ such that $x \in n$ holds $x$ is a natural number.

(5) For every non empty natural number $x$ holds $0 \in x$.

Let us observe that every set which is natural is also cardinal.

Let $X$ be a set. Observe that $\delta_X$ is one-to-one.

One can prove the following proposition

(6) For every set $X$ holds $\overline{\overline{\mathrm{id}_X}} = \overline{\overline{X}}$.

Let $R$ be a trivial binary relation. One can check that $\mathrm{dom} R$ is trivial.

Let us observe that every function which is trivial is also one-to-one.

One can prove the following propositions:

(7) For all functions $f$, $g$ such that $\mathrm{dom} f$ misses $\mathrm{dom} g$ holds $\mathrm{rng}(f+\cdot g) = \mathrm{rng} f \cup \mathrm{rng} g$.

(8) For all one-to-one functions $f$, $g$ such that $\operatorname{dom} f$ misses $\operatorname{dom} g$ and $\operatorname{rng} f$ misses $\operatorname{rng} g$ holds $(f {+\cdot} g)^{-1} = f^{-1} {+\cdot} g^{-1}$.

(9) For all sets $A$, $a$, $b$ holds $(A \longmapsto a) {+\cdot} (A \longmapsto b) = A \longmapsto b$.

(10) For all sets $a$, $b$ holds $(a \dot{\longmapsto} b)^{-1} = b \dot{\longmapsto} a$.

(11) For all sets $a, b, c, d$ such that $a = b$ iff $c = d$ holds $[a \longmapsto c, b \longmapsto d]^{-1} = [c \longmapsto a, d \longmapsto b]$.

The scheme *Convers* deals with a non empty set $\mathcal{A}$, a binary relation $\mathcal{B}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{B}^{\smile} = \{\langle \mathcal{F}(x), \mathcal{G}(x) \rangle; x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\}$

provided the following condition is met:

- $\mathcal{B} = \{\langle \mathcal{G}(x), \mathcal{F}(x) \rangle; x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\}$.

Next we state the proposition

(12) For all natural numbers $i$, $j$, $n$ such that $i < j$ and $j \in n$ holds $i \in n$.

## 2. Auxiliary Concepts

Let $R$, $S$ be relational structures. We say that $S$ embeds $R$ if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a map $f$ from $R$ into $S$ such that

(i) $f$ is one-to-one, and

(ii) for all elements $x$, $y$ of $R$ holds $\langle x, y \rangle \in$ the internal relation of $R$ iff $\langle f(x), f(y) \rangle \in$ the internal relation of $S$.

Let $R$, $S$ be non empty relational structures. Let us note that the predicate $S$ embeds $R$ is reflexive. We now state the proposition

(13) For all non empty relational structures $R$, $S$, $T$ such that $R$ embeds $S$ and $S$ embeds $T$ holds $R$ embeds $T$.

Let $R$, $S$ be non empty relational structures. We say that $R$ is equimorphic to $S$ if and only if:

(Def. 3) $R$ embeds $S$ and $S$ embeds $R$.

Let us notice that the predicate $R$ is equimorphic to $S$ is reflexive and symmetric.

Next we state the proposition

(14) Let $R$, $S$, $T$ be non empty relational structures. Suppose $R$ is equimorphic to $S$ and $S$ is equimorphic to $T$. Then $R$ is equimorphic to $T$.

Let $R$ be a non empty relational structure. We introduce $R$ is parallel as an antonym of $R$ is connected.

Let $R$ be a relational structure. We say that $R$ is symmetric if and only if:

(Def. 4) The internal relation of $R$ is symmetric in the carrier of $R$.

Let $R$ be a relational structure. We say that $R$ is asymmetric if and only if:

(Def. 5) The internal relation of $R$ is asymmetric.

Next we state the proposition

(15) Let $R$ be a relational structure. Suppose $R$ is asymmetric. Then the internal relation of $R$ misses (the internal relation of $R)^{\smile}$.

Let $R$ be a relational structure. We say that $R$ is irreflexive if and only if:

(Def. 6) For every set $x$ such that $x \in$ the carrier of $R$ holds $\langle x, x\rangle \notin$ the internal relation of $R$.

Let $n$ be a natural number. The functor $n$-SuccRelStr yields a strict relational structure and is defined as follows:

(Def. 7) The carrier of $n$-SuccRelStr $= n$ and the internal relation of $n$-SuccRelStr $= \{\langle i, i+1\rangle; i$ ranges over natural numbers: $i+1 < n\}$.

The following two propositions are true:

(16) For every natural number $n$ holds $n$-SuccRelStr is asymmetric.

(17) If $n > 0$, then $\overline{\overline{\text{the internal relation of } n\text{-SuccRelStr}}} = n-1$.

Let $R$ be a relational structure. The functor SymRelStr$R$ yielding a strict relational structure is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of SymRelStr$R =$ the carrier of $R$, and

(ii) the internal relation of SymRelStr$R =$ (the internal relation of $R$) $\cup$ (the internal relation of $R)^{\smile}$.

Let $R$ be a relational structure. One can check that SymRelStr$R$ is symmetric.

Let us note that there exists a relational structure which is non empty and symmetric.

Let $R$ be a symmetric relational structure. Observe that the internal relation of $R$ is symmetric.

Let $R$ be a relational structure. The functor ComplRelStr$R$ yielding a strict relational structure is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of ComplRelStr$R =$ the carrier of $R$, and

(ii) the internal relation of ComplRelStr$R =$ (the internal relation of $R)^{\rm c} \setminus {\rm id}_{\text{the carrier of } R}$.

Let $R$ be a non empty relational structure. Observe that ComplRelStr$R$ is non empty.

Next we state the proposition

(18) Let $S$, $R$ be relational structures. Suppose $S$ and $R$ are isomorphic. Then $\overline{\overline{\text{the internal relation of } S}} = \overline{\overline{\text{the internal relation of } R}}$.

## 3. NECKLACE $n$

Let $n$ be a natural number. The functor Necklace$n$ yielding a strict relational structure is defined by:

(Def. 10) Necklace$n =$ SymRelStr$n$-SuccRelStr.

Let $n$ be a natural number. Observe that Necklace$n$ is symmetric.

We now state two propositions:

(19) The internal relation of Necklace$n = \{\langle i, i+1\rangle; i$ ranges over natural numbers: $i+1 < n\} \cup \{\langle i+1, i\rangle; i$ ranges over natural numbers: $i+1 < n\}$.

(20) Let $x$ be a set. Then $x \in$ the internal relation of Necklace$n$ if and only if there exists a natural number $i$ such that $i+1 < n$ but $x = \langle i, i+1\rangle$ or $x = \langle i+1, i\rangle$.

Let $n$ be a natural number. One can verify that Necklace$n$ is irreflexive.

One can prove the following proposition

(21) For every natural number $n$ holds the carrier of Necklace$n = n$.

Let $n$ be a non empty natural number. Observe that Necklace$n$ is non empty.

Let $n$ be a natural number. Note that the carrier of Necklace$n$ is finite.

The following propositions are true:

(22) For all natural numbers $n$, $i$ such that $i+1<n$ holds $\langle i, i+1\rangle \in$ the internal relation of Necklace $n$.

(23) For every natural number $n$ and for every natural number $i$ such that $i \in$ the carrier of Necklace $n$ holds $i<n$.

(24) For every non empty natural number $n$ holds Necklace $n$ is connected.

(25) For all natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the internal relation of Necklace $n$ holds $i=j+1$ or $j=i+1$.

(26) Let $i$, $j$ be natural numbers. Suppose $i=j+1$ or $j=i+1$ but $i \in$ the carrier of Necklace $n$ but $j \in$ the carrier of Necklace $n$. Then $\langle i, j\rangle \in$ the internal relation of Necklace $n$.

(27) If $n>0$, then $\overline{\overline{\{\langle i+1, i\rangle; i \text{ ranges over natural numbers: } i+1<n\}}} = n-1$.

(28) If $n>0$, then $\overline{\overline{\text{the internal relation of Necklace}\,n}} = 2\cdot(n-1)$.

(29) Necklace 1 and ComplRelStr Necklace 1 are isomorphic.

(30) Necklace 4 and ComplRelStr Necklace 4 are isomorphic.

(31) If Necklace $n$ and ComplRelStr Necklace $n$ are isomorphic, then $n=0$ or $n=1$ or $n=4$.

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