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Natural transformations. Discrete categories

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Summary. We present well known concepts of category theory: natural transformations and functor categories, and prove propositions related to. Because of the formalization it proved to be convenient to introduce some auxiliary notions, for instance: transformations. We mean by a transformation of a functor F to a functor G, both covariant functors from A to B, a function mapping the objects of A to the morphisms of B and assigning to an object a of A an element of $\mathrm{Hom}(F(a), G(a))$. The material included roughly corresponds to that presented on pages 18,129–130,137–138 of the monography ([9]). We also introduce discrete categories and prove some propositions to illustrate the concepts introduced.

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The articles [10], [5], [12], [11], [8], [13], [2], [3], [6], [1], [4], and [7] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we follow the rules: A_1, A_2, B_1, B_2 denote non empty sets, f denotes a function from A_1 into B_1, g denotes a function from A_2 into B_2, Y_1 denotes a non empty subset of A_1, and Y_2 denotes a non empty subset of A_2.

Let A_1 be a set, let B_1 be a non empty set, let f be a function from A_1 into B_1, and let Y_1 be a subset of A_1. Then $f \restriction Y_1$ is a function from Y_1 into B_1.

One can prove the following proposition

(1) $[: f, g :] \restriction [: Y_1, Y_2 :] = [: f \restriction Y_1, g \restriction Y_2 :]$.

Let A, B be non empty sets, let A_1 be a non empty subset of A, let B_1 be a non empty subset of B, let f be a partial function from $[: A_1, A_1 :]$ to A_1, and let g be a partial function from $[: B_1, B_1 :]$ to B_1. Then $|: f, g :|$ is a partial function from $[: [: A_1, B_1 :], [: A_1, B_1 :] :]$ to $[: A_1, B_1 :]$.

The following proposition is true

(2) Let f be a partial function from $[: A_1, A_1 :]$ to A_1, g be a partial function from $[: A_2, A_2 :]$ to A_2, and F be a partial function from $[: Y_1, Y_1 :]$ to Y_1. Suppose $F = f \restriction ([: Y_1, Y_1 :]$ **qua** set$)$. Let G be a partial function from $[: Y_2, Y_2 :]$ to Y_2. If $G = g \restriction ([: Y_2, Y_2 :]$ **qua** set$)$, then $|: F, G :| = |: f, g :| \restriction ([: [: Y_1, Y_2 :], [: Y_1, Y_2 :] :]$ **qua** set$)$.

We adopt the following convention: A, B, C are categories, F, F_1, F_2, F_3 are functors from A to B, and G is a functor from B to C.

The scheme *M Choice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function t from $\mathcal{A}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds $t(a) \in \mathcal{F}(a)$

provided the parameters satisfy the following condition:

- For every element a of $\mathcal{A}$ holds $\mathcal{B}$ meets $\mathcal{F}(a)$.

The following proposition is true

(3) For every object a of A and for every morphism m from a to a holds $m \in \hom(a,a)$.

In the sequel m, o denote sets.

One can prove the following propositions:

(4) For all morphisms f, g of $\circlearrowright(o,m)$ holds $f = g$.

(5) For every object a of A holds $\langle\langle \mathrm{id}_a, \mathrm{id}_a\rangle, \mathrm{id}_a\rangle \in$ the composition of A.

(6) The composition of $\circlearrowright(o,m) = \{\langle\langle m, m\rangle, m\rangle\}$.

(7) For every object a of A holds $\circlearrowright(a, \mathrm{id}_a)$ is a subcategory of A.

(8) Let C be a subcategory of A. Then

(i) the dom-map of $C =$ (the dom-map of A)$\restriction$(the morphisms of C),

(ii) the cod-map of $C =$ (the cod-map of A)$\restriction$(the morphisms of C),

(iii) the composition of $C =$ (the composition of A)$\restriction$[:the morphisms of C, the morphisms of C:], and

(iv) the id-map of $C =$ (the id-map of A)$\restriction$(the objects of C).

(9) Let O be a non empty subset of the objects of A, M be a non empty subset of the morphisms of A, and D_1, C_1 be functions from M into O. Suppose $D_1 =$ (the dom-map of A)$\restriction M$ and $C_1 =$ (the cod-map of A)$\restriction M$. Let C_2 be a partial function from [:M, (M **qua** non empty set):] to M. Suppose $C_2 =$ (the composition of A)$\restriction$([:M, M:] **qua** set). Let I_1 be a function from O into M. Suppose $I_1 =$ (the id-map of A)$\restriction O$. Then $\langle O, M, D_1, C_1, C_2, I_1\rangle$ is a subcategory of A.

(10) Let C be a strict category and A be a strict subcategory of C. Suppose the objects of $A =$ the objects of C and the morphisms of $A =$ the morphisms of C. Then $A = C$.

2. Application of a functor to a morphism

Let us consider A, B, F and let a, b be objects of A. Let us assume that $\hom(a,b) \neq \emptyset$. Let f be a morphism from a to b. The functor $F(f)$ yielding a morphism from $F(a)$ to $F(b)$ is defined as follows:

(Def. 1) $F(f) = F(f)$.

One can prove the following propositions:

(11) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b holds $(G \cdot F)(f) = G(F(f))$.

(12) Let F_1, F_2 be functors from A to B. Suppose that for all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b holds $F_1(f) = F_2(f)$. Then $F_1 = F_2$.

(13) Let a, b, c be objects of A. Suppose $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $F(g \cdot f) = F(g) \cdot F(f)$.

(14) For every object c of A and for every object d of B such that $F(\mathrm{id}_c) = \mathrm{id}_d$ holds $F(c) = d$.

(15) For every object a of A holds $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

(16) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b holds $\mathrm{id}_A(f) = f$.

(17) For all objects a, b, c, d of A such that $\hom(a,b)$ meets $\hom(c,d)$ holds $a = c$ and $b = d$.

3. Transformations

Let us consider A, B, F_1, F_2. We say that F_1 is transformable to F_2 if and only if:

(Def. 2) For every object a of A holds $\hom(F_1(a), F_2(a)) \neq \emptyset$.

Let us note that the predicate F_1 is transformable to F_2 is reflexive.

We now state the proposition

(19)[1] If F is transformable to F_1 and F_1 is transformable to F_2, then F is transformable to F_2.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. A function from the objects of A into the morphisms of B is said to be a transformation from F_1 to F_2 if:

(Def. 3) For every object a of A holds $\mathrm{it}(a)$ is a morphism from $F_1(a)$ to $F_2(a)$.

Let us consider A, B and let F be a functor from A to B. The functor id_F yields a transformation from F to F and is defined by:

(Def. 4) For every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2 and let a be an object of A. The functor $t(a)$ yielding a morphism from $F_1(a)$ to $F_2(a)$ is defined as follows:

(Def. 5) $t(a) = t(a)$.

Let us consider A, B, F, F_1, F_2. Let us assume that F is transformable to F_1 and F_1 is transformable to F_2. Let t_1 be a transformation from F to F_1 and let t_2 be a transformation from F_1 to F_2. The functor $t_2 \circ t_1$ yields a transformation from F to F_2 and is defined by:

(Def. 6) For every object a of A holds $(t_2 \circ t_1)(a) = t_2(a) \cdot t_1(a)$.

We now state four propositions:

(20) Suppose F_1 is transformable to F_2. Let t_1, t_2 be transformations from F_1 to F_2. If for every object a of A holds $t_1(a) = t_2(a)$, then $t_1 = t_2$.

(21) For every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

(22) If F_1 is transformable to F_2, then for every transformation t from F_1 to F_2 holds $\mathrm{id}_{(F_2)} \circ t = t$ and $t \circ \mathrm{id}_{(F_1)} = t$.

(23) Suppose F is transformable to F_1 and F_1 is transformable to F_2 and F_2 is transformable to F_3. Let t_1 be a transformation from F to F_1, t_2 be a transformation from F_1 to F_2, and t_3 be a transformation from F_2 to F_3. Then $(t_3 \circ t_2) \circ t_1 = t_3 \circ (t_2 \circ t_1)$.

4. Natural transformations

Let us consider A, B, F_1, F_2. We say that F_1 is naturally transformable to F_2 if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) F_1 is transformable to F_2, and

(ii) there exists a transformation t from F_1 to F_2 such that for all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b holds $t(b) \cdot F_1(f) = F_2(f) \cdot t(a)$.

Let us note that the predicate F_1 is naturally transformable to F_2 is reflexive.

One can prove the following proposition

[1] The proposition (18) has been removed.

(25)[2] Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Then F is naturally transformable to F_2.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is naturally transformable to F_2. A transformation from F_1 to F_2 is said to be a natural transformation from F_1 to F_2 if:

(Def. 8) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b holds $\mathrm{it}(b) \cdot F_1(f) = F_2(f) \cdot \mathrm{it}(a)$.

Let us consider A, B, F. Then id_F is a natural transformation from F to F.

Let us consider A, B, F, F_1, F_2. Let us assume that F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1 and let t_2 be a natural transformation from F_1 to F_2. The functor $t_2 \circ t_1$ yields a natural transformation from F to F_2 and is defined by:

(Def. 9) $t_2 \circ t_1 = t_2 \circ t_1$.

Next we state the proposition

(26) Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathrm{id}_{(F_2)} \circ t = t$ and $t \circ \mathrm{id}_{(F_1)} = t$.

In the sequel t denotes a natural transformation from F to F_1 and t_1 denotes a natural transformation from F_1 to F_2.

One can prove the following propositions:

(27) Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1, t_2 be a natural transformation from F_1 to F_2, and a be an object of A. Then $(t_2 \circ t_1)(a) = t_2(a) \cdot t_1(a)$.

(28) Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3. Let t_3 be a natural transformation from F_2 to F_3. Then $(t_3 \circ t_1) \circ t = t_3 \circ (t_1 \circ t)$.

Let us consider A, B, F_1, F_2 and let I_2 be a transformation from F_1 to F_2. We say that I_2 is invertible if and only if:

(Def. 10) For every object a of A holds $I_2(a)$ is invertible.

Let us consider A, B, F_1, F_2. We say that F_1 and F_2 are naturally equivalent if and only if:

(Def. 11) F_1 is naturally transformable to F_2 and there exists a natural transformation from F_1 to F_2 which is invertible.

Let us note that the predicate F_1 and F_2 are naturally equivalent is reflexive. We introduce $F_1 \cong F_2$ as a synonym of F_1 and F_2 are naturally equivalent.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. Let t_1 be a transformation from F_1 to F_2. Let us assume that t_1 is invertible. The functor $t_1{}^{-1}$ yielding a transformation from F_2 to F_1 is defined as follows:

(Def. 12) For every object a of A holds $t_1{}^{-1}(a) = t_1(a)^{-1}$.

Let us consider A, B, F_1, F_2, t_1. Let us assume that F_1 is naturally transformable to F_2 and t_1 is invertible. The functor $t_1{}^{-1}$ yields a natural transformation from F_2 to F_1 and is defined by:

(Def. 13) $t_1{}^{-1} = (t_1$ **qua** transformation from F_1 to $F_2)^{-1}$.

The following three propositions are true:

[2] The proposition (24) has been removed.

(30)[3] Let given A, B, F_1, F_2, t_1. Suppose F_1 is naturally transformable to F_2 and t_1 is invertible. Let a be an object of A. Then ${t_1}^{-1}(a) = t_1(a)^{-1}$.

(31) If $F_1 \cong F_2$, then $F_2 \cong F_1$.

(32) If $F_1 \cong F_2$ and $F_2 \cong F_3$, then $F_1 \cong F_3$.

Let us consider A, B, F_1, F_2. Let us assume that F_1 and F_2 are naturally equivalent. A natural transformation from F_1 to F_2 is said to be a natural equivalence of F_1 and F_2 if:

(Def. 14) It is invertible.

The following propositions are true:

(33) id_F is a natural equivalence of F and F.

(34) Suppose $F_1 \cong F_2$ and $F_2 \cong F_3$. Let t be a natural equivalence of F_1 and F_2 and t' be a natural equivalence of F_2 and F_3. Then $t' \circ t$ is a natural equivalence of F_1 and F_3.

5. Functor category

Let us consider A, B. A non empty set is called a set of natural transformations from A to B if it satisfies the condition (Def. 15).

(Def. 15) Let x be a set. Suppose $x \in$ it. Then there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that $x = \langle\langle F_1, F_2\rangle, t\rangle$ and F_1 is naturally transformable to F_2.

Let us consider A, B. The functor $\mathrm{NatTrans}(A,B)$ yielding a set of natural transformations from A to B is defined by the condition (Def. 16).

(Def. 16) Let x be a set. Then $x \in \mathrm{NatTrans}(A,B)$ if and only if there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that $x = \langle\langle F_1, F_2\rangle, t\rangle$ and F_1 is naturally transformable to F_2.

Let A_1, B_1, A_2, B_2 be non empty sets, let f_1 be a function from A_1 into B_1, and let f_2 be a function from A_2 into B_2. Let us observe that $f_1 = f_2$ if and only if:

(Def. 17) $A_1 = A_2$ and for every element a of A_1 holds $f_1(a) = f_2(a)$.

Next we state the proposition

(35) F_1 is naturally transformable to F_2 iff $\langle\langle F_1, F_2\rangle, t_1\rangle \in \mathrm{NatTrans}(A,B)$.

Let us consider A, B. The functor B^A yielding a strict category is defined by the conditions (Def. 18).

(Def. 18)(i) The objects of $B^A = \mathrm{Funct}(A,B)$,

(ii) the morphisms of $B^A = \mathrm{NatTrans}(A,B)$,

(iii) for every morphism f of B^A holds $\mathrm{dom} f = (f_{\mathbf{1}})_{\mathbf{1}}$ and $\mathrm{cod} f = (f_{\mathbf{1}})_{\mathbf{2}}$,

(iv) for all morphisms f, g of B^A such that $\mathrm{dom}\, g = \mathrm{cod} f$ holds $\langle g, f\rangle \in \mathrm{dom}(\text{the composition of } B^A)$,

(v) for all morphisms f, g of B^A such that $\langle g, f\rangle \in \mathrm{dom}(\text{the composition of } B^A)$ there exist F, F_1, F_2, t, t_1 such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_1, F_2\rangle, t_1\rangle$ and (the composition of B^A)($\langle g, f\rangle) = \langle\langle F, F_2\rangle, t_1 \circ t\rangle$, and

(vi) for every object a of B^A and for every F such that $F = a$ holds $\mathrm{id}_a = \langle\langle F, F\rangle, \mathrm{id}_F\rangle$.

The following propositions are true:

[3] The proposition (29) has been removed.

(39)[4] For every morphism f of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ holds $\mathrm{dom} f = F$ and $\mathrm{cod} f = F_1$.

(40) Let a, b be objects of B^A and f be a morphism from a to b. If $\hom(a,b) \neq \emptyset$, then there exist F, F_1, t such that $a = F$ and $b = F_1$ and $f = \langle\langle F, F_1\rangle, t\rangle$.

(41) Let t' be a natural transformation from F_2 to F_3 and f, g be morphisms of B^A. Suppose $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_2, F_3\rangle, t'\rangle$. Then $\langle g, f\rangle \in \mathrm{dom}$(the composition of B^A) if and only if $F_1 = F_2$.

(42) For all morphisms f, g of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_1, F_2\rangle, t_1\rangle$ holds $g \cdot f = \langle\langle F, F_2\rangle, t_1 \circ t\rangle$.

6. Discrete categories

Let C be a category. We say that C is discrete if and only if:

(Def. 19) For every morphism f of C there exists an object a of C such that $f = \mathrm{id}_a$.

One can verify that there exists a category which is discrete.
The following propositions are true:

(44)[5] For every discrete category A and for every object a of A holds $\hom(a,a) = \{\mathrm{id}_a\}$.

(45) A is discrete if and only if the following conditions are satisfied:

(i) for every object a of A there exists a finite set B such that $B = \hom(a,a)$ and $\mathrm{card} B = 1$, and

(ii) for all objects a, b of A such that $a \neq b$ holds $\hom(a,b) = \emptyset$.

(46) $\circlearrowleft(o,m)$ is discrete.

(47) For every discrete category A holds every subcategory of A is discrete.

(48) If A is discrete and B is discrete, then $[: A, B :]$ is discrete.

(49) Let A be a discrete category, B be a category, and F_1, F_2 be functors from B to A. If F_1 is transformable to F_2, then $F_1 = F_2$.

(50) Let A be a discrete category, B be a category, F be a functor from B to A, and t be a transformation from F to F. Then $t = \mathrm{id}_F$.

(51) If A is discrete, then A^B is discrete.

Let C be a category. Note that there exists a subcategory of C which is strict and discrete.
Let us consider C. The functor $\mathrm{IdCat}\, C$ yields a strict discrete subcategory of C and is defined by:

(Def. 20) The objects of $\mathrm{IdCat}\, C$ = the objects of C and the morphisms of $\mathrm{IdCat}\, C = \{\mathrm{id}_a : a$ ranges over objects of $C\}$.

One can prove the following propositions:

(52) For every strict category C such that C is discrete holds $\mathrm{IdCat}\, C = C$.

(53) $\mathrm{IdCat}\,\mathrm{IdCat}\, C = \mathrm{IdCat}\, C$.

(54) $\mathrm{IdCat} \circlearrowleft(o,m) = \circlearrowleft(o,m)$.

(55) $\mathrm{IdCat}[: A, B :] = [: \mathrm{IdCat}\, A, \mathrm{IdCat}\, B :]$.

[4] The propositions (36)–(38) have been removed.
[5] The proposition (43) has been removed.

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