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The Correspondence Between Homomorphisms of Universal Algebra & Many Sorted Algebra

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Summary. The aim of the article is to check the compatibility of the homomorphism of universal algebras introduced in [9] and the corresponding concept for many sorted algebras introduced in [10].

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The articles [14], [18], [19], [4], [20], [6], [5], [15], [1], [7], [11], [2], [3], [12], [8], [17], [9], [10], [13], and [16] provide the notation and terminology for this paper.

For simplicity, we follow the rules: U_1, U_2, U_3 are universal algebras, n is a natural number, A is a non empty set, and h is a function from U_1 into U_2.

Next we state four propositions:

(1) For all functions f, g and for every set C such that $\operatorname{rng} f \subseteq C$ holds $(g \restriction C) \cdot f = g \cdot f$.

(2) For every set I and for every subset C of I holds $C^* \subseteq I^*$.

(3) For every function f and for every set C such that f is function yielding holds $f \restriction C$ is function yielding.

(4) For every set I and for every subset C of I and for every many sorted set M indexed by I holds $(M \restriction C)^\# = M^\# \restriction C^*$.

Let us consider A, n and let a be an element of A. Then $n \mapsto a$ is a finite sequence of elements of A.

Let S, S' be non empty many sorted signatures. The predicate $S \leq S'$ is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $S \subseteq$ the carrier of S',

(ii) the operation symbols of $S \subseteq$ the operation symbols of S',

(iii) (the arity of S')$\restriction$(the operation symbols of S) $=$ the arity of S, and

(iv) (the result sort of S')$\restriction$(the operation symbols of S) $=$ the result sort of S.

Let us note that the predicate $S \leq S'$ is reflexive.

One can prove the following propositions:

(5) For all non empty many sorted signatures S, S', S'' such that $S \leq S'$ and $S' \leq S''$ holds $S \leq S''$.

(6) For all strict non empty many sorted signatures S, S' such that $S \leq S'$ and $S' \leq S$ holds $S = S'$.

(7) Let g be a function, a be an element of A, and k be a natural number. If $1 \leq k$ and $k \leq n$, then $(a \dot{\longmapsto} g)((n \mapsto a)_k) = g$.

(8) Let I be a set, I_0 be a subset of I, A, B be many sorted sets indexed by I, F be a many sorted function from A into B, and A_0, B_0 be many sorted sets indexed by I_0. Suppose $A_0 = A \restriction I_0$ and $B_0 = B \restriction I_0$. Then $F \restriction I_0$ is a many sorted function from A_0 into B_0.

Let S, S' be strict non void non empty many sorted signatures and let A be a non-empty strict algebra over S'. Let us assume that $S \leq S'$. The functor $(A \operatorname{over} S)$ yields a non-empty strict algebra over S and is defined by the conditions (Def. 2).

(Def. 2)(i) The sorts of $(A \operatorname{over} S) =$ (the sorts of A)$\restriction$(the carrier of S), and

(ii) the characteristics of $(A \operatorname{over} S) =$ (the characteristics of A)$\restriction$(the operation symbols of S).

One can prove the following propositions:

(9) For every strict non void non empty many sorted signature S and for every non-empty strict algebra A over S holds $A = (A \operatorname{over} S)$.

(10) For all U_1, U_2 such that U_1 and U_2 are similar holds $\mathrm{MSSign}(U_1) = \mathrm{MSSign}(U_2)$.

Let U_1, U_2 be universal algebras and let h be a function from U_1 into U_2. Let us assume that $\mathrm{MSSign}(U_1) = \mathrm{MSSign}(U_2)$. The functor $\mathrm{MSAlg}(h)$ yields a many sorted function from $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \operatorname{over} \mathrm{MSSign}(U_1))$ and is defined as follows:

(Def. 3) $\mathrm{MSAlg}(h) = \{0\} \longmapsto h$.

We now state a number of propositions:

(11) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\mathrm{MSSign}(U_1)$. Then $(\mathrm{MSAlg}(h))$(the result sort of o) $= h$.

(12) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds $\mathrm{Den}(o, \mathrm{MSAlg}(U_1)) =$ (the characteristic of U_1)(o).

(13) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds $\mathrm{Den}(o, \mathrm{MSAlg}(U_1))$ is an operation of U_1.

(14) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds every element of $\mathrm{Args}(o, \mathrm{MSAlg}(U_1))$ is a finite sequence of elements of the carrier of U_1.

(15) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\mathrm{MSSign}(U_1)$ and y be an element of $\mathrm{Args}(o, \mathrm{MSAlg}(U_1))$. Then $\mathrm{MSAlg}(h)\#y = h \cdot y$.

(16) If h is a homomorphism of U_1 into U_2, then $\mathrm{MSAlg}(h)$ is a homomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \operatorname{over} \mathrm{MSSign}(U_1))$.

(17) If U_1 and U_2 are similar, then $\mathrm{MSAlg}(h)$ is a many sorted set indexed by $\{0\}$.

(18) If h is an epimorphism of U_1 onto U_2, then $\mathrm{MSAlg}(h)$ is an epimorphism of $\mathrm{MSAlg}(U_1)$ onto $(\mathrm{MSAlg}(U_2) \operatorname{over} \mathrm{MSSign}(U_1))$.

(19) If h is a monomorphism of U_1 into U_2, then $\mathrm{MSAlg}(h)$ is a monomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \operatorname{over} \mathrm{MSSign}(U_1))$.

(20) If h is an isomorphism of U_1 and U_2, then $\mathrm{MSAlg}(h)$ is an isomorphism of $\mathrm{MSAlg}(U_1)$ and $(\mathrm{MSAlg}(U_2) \operatorname{over} \mathrm{MSSign}(U_1))$.

(21) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is a homomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is a homomorphism of U_1 into U_2.

(22) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is an epimorphism of $\mathrm{MSAlg}(U_1)$ onto $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is an epimorphism of U_1 onto U_2.

(23) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is a monomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is a monomorphism of U_1 into U_2.

(24) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is an isomorphism of $\mathrm{MSAlg}(U_1)$ and $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is an isomorphism of U_1 and U_2.

(25) $\mathrm{MSAlg}(\mathrm{id}_{\text{the carrier of } U_1}) = \mathrm{id}_{\text{the sorts of } \mathrm{MSAlg}(U_1)}$.

(26) Let given U_1, U_2, U_3. Suppose U_1 and U_2 are similar and U_2 and U_3 are similar. Let h_1 be a function from U_1 into U_2 and h_2 be a function from U_2 into U_3. Then $\mathrm{MSAlg}(h_2) \circ \mathrm{MSAlg}(h_1) = \mathrm{MSAlg}(h_2 \cdot h_1)$.

REFERENCES

[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Ewa Burakowska. Subalgebras of the universal algebra. Lattices of subalgebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/unialg_2.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[7] Czesław Byliński. A classical first order language. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/cqc_lang.html.

[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[9] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/alg_1.html.

[10] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[11] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/unialg_1.html.

[12] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[13] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pre_circ.html.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[15] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[16] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[17] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[19] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[20] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

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