JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Homomorphisms of Many Sorted Algebras

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Summary. The aim of this article is to present the definition and some properties of homomorphisms of many sorted algebras. Some auxiliary properties of many sorted functions also have been shown.

MML Identifier: MSUALG_3.

WWW: http://mizar.org/JFM/Vol6/msualg_3.html

The articles [11], [15], [16], [6], [8], [7], [4], [2], [14], [1], [3], [12], [9], [13], [5], and [10] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we use the following convention: S is a non void non empty many sorted signature, U_1, U_2 are algebras over S, o is an operation symbol of S, and n is a natural number.

Let I be a non empty set, let A, B be many sorted sets indexed by I, let F be a many sorted function from A into B, and let i be an element of I. Then $F(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider S and let U_1, U_2 be algebras over S. A many sorted function from U_1 into U_2 is a many sorted function from the sorts of U_1 into the sorts of U_2.

Let I be a set and let A be a many sorted set indexed by I. The functor id_A yields a many sorted function from A into A and is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $\mathrm{id}_A(i) = \mathrm{id}_{A(i)}$.

Let I_1 be a function. We say that I_1 is "1-1" if and only if:

(Def. 2) For every set i and for every function f such that $i \in \mathrm{dom} I_1$ and $I_1(i) = f$ holds f is one-to-one.

Let I be a set. Observe that there exists a many sorted function indexed by I which is "1-1".

Next we state the proposition

(1) Let I be a set and F be a many sorted function indexed by I. Then F is "1-1" if and only if for every set i such that $i \in I$ holds $F(i)$ is one-to-one.

Let I be a set, let A, B be many sorted sets indexed by I, and let I_1 be a many sorted function from A into B. We say that I_1 is onto if and only if:

(Def. 3) For every set i such that $i \in I$ holds $\mathrm{rng} I_1(i) = B(i)$.

Let F, G be function yielding functions. The functor $G \circ F$ yields a function yielding function and is defined as follows:

(Def. 4) $\mathrm{dom}(G \circ F) = \mathrm{dom}\, F \cap \mathrm{dom}\, G$ and for every set i such that $i \in \mathrm{dom}(G \circ F)$ holds $(G \circ F)(i) = G(i) \cdot F(i)$.

One can prove the following proposition

(2) Let I be a set, A, B, C be many sorted sets indexed by I, F be a many sorted function from A into B, and G be a many sorted function from B into C. Then $\mathrm{dom}(G \circ F) = I$ and for every set i such that $i \in I$ holds $(G \circ F)(i) = G(i) \cdot F(i)$.

Let I be a set, let A be a many sorted set indexed by I, let B, C be non-empty many sorted sets indexed by I, let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

One can prove the following propositions:

(3) Let I be a set, A, B be many sorted sets indexed by I, and F be a many sorted function from A into B. Then $F \circ \mathrm{id}_A = F$.

(4) Let I be a set, A, B be many sorted sets indexed by I, and F be a many sorted function from A into B. Then $\mathrm{id}_B \circ F = F$.

Let I be a set, let A, B be many sorted sets indexed by I, and let F be a many sorted function from A into B. Let us assume that F is "1-1" and onto. The functor F^{-1} yielding a many sorted function from B into A is defined by:

(Def. 5) For every set i such that $i \in I$ holds $F^{-1}(i) = F(i)^{-1}$.

Next we state the proposition

(5) Let I be a set, A, B be non-empty many sorted sets indexed by I, H be a many sorted function from A into B, and H_1 be a many sorted function from B into A. If H is "1-1" and onto and $H_1 = H^{-1}$, then $H \circ H_1 = \mathrm{id}_B$ and $H_1 \circ H = \mathrm{id}_A$.

Let I be a set, let A be a many sorted set indexed by I, and let F be a many sorted function indexed by I. The functor $F^{\circ} A$ yields a many sorted set indexed by I and is defined as follows:

(Def. 6) For every set i such that $i \in I$ holds $(F^{\circ} A)(i) = F(i)^{\circ} A(i)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let us consider o. One can check that every element of $\mathrm{Args}(o, U_1)$ is function-like and relation-like.

2. Homomorphisms of Many Sorted Algebras

We now state the proposition

(6) Let U_1 be an algebra over S and x be a function. Suppose $x \in \mathrm{Args}(o, U_1)$. Then $\mathrm{dom}\, x = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every set y such that $y \in \mathrm{dom}((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))$ holds $x(y) \in ((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))(y)$.

Let us consider S, let U_1, U_2 be algebras over S, let us consider o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. Let us assume that $\mathrm{Args}(o, U_1) \neq \emptyset$ and $\mathrm{Args}(o, U_2) \neq \emptyset$. The functor $F\#x$ yields an element of $\mathrm{Args}(o, U_2)$ and is defined as follows:

(Def. 7) $F\#x = (\mathrm{Frege}(F \cdot \mathrm{Arity}(o)))(x)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let us consider o. One can verify that there exists an element of $\mathrm{Args}(o, U_1)$ which is function-like and relation-like.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let us consider o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. Then $F\#x$ is a function-like relation-like element of $\mathrm{Args}(o, U_2)$ and it can be characterized by the condition:

(Def. 8) For every n such that $n \in \mathrm{dom}\, x$ holds $(F\#x)(n) = F(\mathrm{Arity}(o)_n)(x(n))$.

We now state two propositions:

(7) Let given S, o and U_1 be an algebra over S. If $\mathrm{Args}(o,U_1) \neq \emptyset$, then for every element x of $\mathrm{Args}(o,U_1)$ holds $x = \mathrm{id}_{\text{the sorts of } U_1}\#x$.

(8) Let U_1, U_2, U_3 be non-empty algebras over S, H_1 be a many sorted function from U_1 into U_2, H_2 be a many sorted function from U_2 into U_3, and x be an element of $\mathrm{Args}(o,U_1)$. Then $(H_2 \circ H_1)\#x = H_2\#(H_1\#x)$.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is a homomorphism of U_1 into U_2 if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let o be an operation symbol of S. Suppose $\mathrm{Args}(o,U_1) \neq \emptyset$. Let x be an element of $\mathrm{Args}(o,U_1)$. Then $F(\text{the result sort of } o)((\mathrm{Den}(o,U_1))(x)) = (\mathrm{Den}(o,U_2))(F\#x)$.

Next we state two propositions:

(9) For every algebra U_1 over S holds $\mathrm{id}_{\text{the sorts of } U_1}$ is a homomorphism of U_1 into U_1.

(10) Let U_1, U_2, U_3 be non-empty algebras over S, H_1 be a many sorted function from U_1 into U_2, and H_2 be a many sorted function from U_2 into U_3. Suppose H_1 is a homomorphism of U_1 into U_2 and H_2 is a homomorphism of U_2 into U_3. Then $H_2 \circ H_1$ is a homomorphism of U_1 into U_3.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is an epimorphism of U_1 onto U_2 if and only if:

(Def. 10) F is a homomorphism of U_1 into U_2 and onto.

One can prove the following proposition

(11) Let U_1, U_2, U_3 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and G be a many sorted function from U_2 into U_3. Suppose F is an epimorphism of U_1 onto U_2 and G is an epimorphism of U_2 onto U_3. Then $G \circ F$ is an epimorphism of U_1 onto U_3.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is a monomorphism of U_1 into U_2 if and only if:

(Def. 11) F is a homomorphism of U_1 into U_2 and "1-1".

The following proposition is true

(12) Let U_1, U_2, U_3 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and G be a many sorted function from U_2 into U_3. Suppose F is a monomorphism of U_1 into U_2 and G is a monomorphism of U_2 into U_3. Then $G \circ F$ is a monomorphism of U_1 into U_3.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is an isomorphism of U_1 and U_2 if and only if:

(Def. 12) F is an epimorphism of U_1 onto U_2 and a monomorphism of U_1 into U_2.

One can prove the following three propositions:

(13) Let F be a many sorted function from U_1 into U_2. Then F is an isomorphism of U_1 and U_2 if and only if F is a homomorphism of U_1 into U_2, onto, and "1-1".

(14) Let U_1, U_2 be non-empty algebras over S, H be a many sorted function from U_1 into U_2, and H_1 be a many sorted function from U_2 into U_1. Suppose H is an isomorphism of U_1 and U_2 and $H_1 = H^{-1}$. Then H_1 is an isomorphism of U_2 and U_1.

(15) Let U_1, U_2, U_3 be non-empty algebras over S, H be a many sorted function from U_1 into U_2, and H_1 be a many sorted function from U_2 into U_3. Suppose H is an isomorphism of U_1 and U_2 and H_1 is an isomorphism of U_2 and U_3. Then $H_1 \circ H$ is an isomorphism of U_1 and U_3.

Let us consider S and let U_1, U_2 be algebras over S. We say that U_1 and U_2 are isomorphic if and only if:

(Def. 13) There exists a many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2.

Next we state the proposition

(16) For every algebra U_1 over S holds $\mathrm{id}_{\text{the sorts of } U_1}$ is an isomorphism of U_1 and U_1 and U_1 and U_1 are isomorphic.

Let us consider S and let U_1, U_2 be algebras over S. Let us note that the predicate U_1 and U_2 are isomorphic is reflexive.

Next we state two propositions:

(17) For all non-empty algebras U_1, U_2 over S such that U_1 and U_2 are isomorphic holds U_2 and U_1 are isomorphic.

(18) Let U_1, U_2, U_3 be non-empty algebras over S. Suppose U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic. Then U_1 and U_3 are isomorphic.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{Im}\,F$ yields a strict non-empty subalgebra of U_2 and is defined as follows:

(Def. 14) The sorts of $\mathrm{Im}\,F = F^{\circ}$(the sorts of U_1).

Next we state several propositions:

(19) Let U_1 be a non-empty algebra over S, U_2 be a strict non-empty algebra over S, and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then F is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im}\,F = U_2$.

(20) Let U_1, U_2 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and G be a many sorted function from U_1 into $\mathrm{Im}\,F$. Suppose $F = G$ and F is a homomorphism of U_1 into U_2. Then G is an epimorphism of U_1 onto $\mathrm{Im}\,F$.

(21) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function G from U_1 into $\mathrm{Im}\,F$ such that $F = G$ and G is an epimorphism of U_1 onto $\mathrm{Im}\,F$.

(22) Let U_1 be a non-empty algebra over S, U_2 be a non-empty subalgebra of U_1, and G be a many sorted function from U_2 into U_1. If $G = \mathrm{id}_{\text{the sorts of } U_2}$, then G is a monomorphism of U_2 into U_1.

(23) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function F_1 from U_1 into $\mathrm{Im}\,F$ and there exists a many sorted function F_2 from $\mathrm{Im}\,F$ into U_2 such that F_1 is an epimorphism of U_1 onto $\mathrm{Im}\,F$ and F_2 is a monomorphism of $\mathrm{Im}\,F$ into U_2 and $F = F_2 \circ F_1$.

(24) Let given S, U_1, U_2 be algebras over S, given o, F be a many sorted function from U_1 into U_2, x be an element of $\mathrm{Args}(o, U_1)$, and f, u be functions. Suppose $x = f$ and $x \in \mathrm{Args}(o, U_1)$ and $u \in \mathrm{Args}(o, U_2)$. Then $u = F\#x$ if and only if for every n such that $n \in \mathrm{dom}\, f$ holds $u(n) = F(\mathrm{Arity}(o)_n)(f(n))$.

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Received April 25, 1994

Published January 2, 2004