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Subalgebras of Many Sorted Algebra. Lattice of Subalgebras

Ewa Burakowska
Warsaw University
Białystok

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The articles [10], [6], [13], [14], [4], [5], [2], [9], [7], [15], [3], [8], [1], [11], and [12] provide the notation and terminology for this paper.

1. Auxiliary Facts about Many Sorted Sets

In this paper x denotes a set.

The scheme *LambdaB* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\mathrm{dom} f = \mathcal{A}$ and for every element d of $\mathcal{A}$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

Let I be a set, let X be a many sorted set indexed by I, and let Y be a non-empty many sorted set indexed by I. One can check that $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

The following proposition is true

(2)[1] Let I be a non empty set, X, Y be many sorted sets indexed by I, and i be an element of I^*. Then $\prod((X \cap Y) \cdot i) = \prod(X \cdot i) \cap \prod(Y \cdot i)$.

Let I be a set and let M be a many sorted set indexed by I. A many sorted set indexed by I is said to be a many sorted subset indexed by M if:

(Def. 1) It $\subseteq M$.

Let I be a set and let M be a non-empty many sorted set indexed by I. One can verify that there exists a many sorted subset indexed by M which is non-empty.

2. Constants of a Many Sorted Algebra

We follow the rules: S is a non void non empty many sorted signature, o is an operation symbol of S, and U_0, U_1, U_2 are algebras over S.

Let S be a non empty many sorted signature and let U_0 be an algebra over S. A subset of U_0 is a many sorted subset indexed by the sorts of U_0.

Let S be a non empty many sorted signature and let I_1 be a sort symbol of S. We say that I_1 has constants if and only if:

[1] The proposition (1) has been removed.

(Def. 2) There exists an operation symbol o of S such that (the arity of $S)(o) = \emptyset$ and (the result sort of $S)(o) = I_1$.

Let I_1 be a non empty many sorted signature. We say that I_1 has constant operations if and only if:

(Def. 3) Every sort symbol of I_1 has constants.

Let A be a non empty set, let B be a set, let a be a function from B into A^*, and let r be a function from B into A. One can check that $\langle A, B, a, r \rangle$ is non empty.

Let us mention that there exists a non empty many sorted signature which is non void and strict and has constant operations.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let s be a sort symbol of S. The functor $\mathrm{Constants}(U_0, s)$ yielding a subset of (the sorts of $U_0)(s)$ is defined by:

(Def. 4)(i) There exists a non empty set A such that $A =$ (the sorts of $U_0)(s)$ and $\mathrm{Constants}(U_0, s) = \{a; a$ ranges over elements of A: $\bigvee_{o\,:\,\text{operation symbol of } S}$ ((the arity of $S)(o) = \emptyset\ \wedge$ (the result sort of $S)(o) = s\ \wedge\ a \in \mathrm{rng}\,\mathrm{Den}(o, U_0))\}$ if (the sorts of $U_0)(s) \neq \emptyset$,

(ii) $\mathrm{Constants}(U_0, s) = \emptyset$, otherwise.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor $\mathrm{Constants}(U_0)$ yields a subset of U_0 and is defined by:

(Def. 5) For every sort symbol s of S holds $(\mathrm{Constants}(U_0))(s) = \mathrm{Constants}(U_0, s)$.

Let S be a non void non empty many sorted signature with constant operations, let U_0 be a non-empty algebra over S, and let s be a sort symbol of S. Observe that $\mathrm{Constants}(U_0, s)$ is non empty.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. One can verify that $\mathrm{Constants}(U_0)$ is non-empty.

3. Subalgebras of a Many Sorted Algebra

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let o be an operation symbol of S, and let A be a subset of U_0. We say that A is closed on o if and only if:

(Def. 6) $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction (A^\# \cdot \text{the arity of } S)(o)) \subseteq (A \cdot \text{the result sort of } S)(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. We say that A is operations closed if and only if:

(Def. 7) For every operation symbol o of S holds A is closed on o.

Next we state the proposition

(3) Let S be a non void non empty many sorted signature, o be an operation symbol of S, U_0 be an algebra over S, and B_0, B_1 be subsets of U_0. If $B_0 \subseteq B_1$, then $({B_0}^\# \cdot \text{the arity of } S)(o) \subseteq ({B_1}^\# \cdot \text{the arity of } S)(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let o be an operation symbol of S, and let A be a subset of U_0. Let us assume that A is closed on o. The functor o_A yielding a function from $(A^\# \cdot \text{the arity of } S)(o)$ into $(A \cdot \text{the result sort of } S)(o)$ is defined by:

(Def. 8) $o_A = \mathrm{Den}(o, U_0) \restriction (A^\# \cdot \text{the arity of } S)(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{Opers}(U_0, A)$ yields a many sorted function from $A^\# \cdot$ the arity of S into $A \cdot$ the result sort of S and is defined as follows:

(Def. 9) For every operation symbol o of S holds $(\mathrm{Opers}(U_0, A))(o) = o_A$.

We now state two propositions:

(4) Let U_0 be an algebra over S and B be a subset of U_0. Suppose $B =$ the sorts of U_0. Then B is operations closed and for every o holds $o_B = \mathrm{Den}(o, U_0)$.

(5) For every subset B of U_0 such that $B =$ the sorts of U_0 holds $\mathrm{Opers}(U_0, B) =$ the characteristics of U_0.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. An algebra over S is called a subalgebra of U_0 if it satisfies the conditions (Def. 10).

(Def. 10)(i) The sorts of it are a subset of U_0, and

(ii) for every subset B of U_0 such that $B =$ the sorts of it holds B is operations closed and the characteristics of it $= \mathrm{Opers}(U_0, B)$.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. Observe that there exists a subalgebra of U_0 which is strict.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. Observe that $\langle$the sorts of U_0, the characteristics of $U_0\rangle$ is non-empty.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. Note that there exists a subalgebra of U_0 which is non-empty and strict.

One can prove the following propositions:

(6) U_0 is a subalgebra of U_0.

(7) If U_0 is a subalgebra of U_1 and U_1 is a subalgebra of U_2, then U_0 is a subalgebra of U_2.

(8) If U_1 is a strict subalgebra of U_2 and U_2 is a strict subalgebra of U_1, then $U_1 = U_2$.

(9) For all subalgebras U_1, U_2 of U_0 such that the sorts of $U_1 \subseteq$ the sorts of U_2 holds U_1 is a subalgebra of U_2.

(10) For all strict subalgebras U_1, U_2 of U_0 such that the sorts of $U_1 =$ the sorts of U_2 holds $U_1 = U_2$.

(11) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, and U_1 be a subalgebra of U_0. Then $\mathrm{Constants}(U_0)$ is a subset of U_1.

(12) Let S be a non void non empty many sorted signature with constant operations, U_0 be a non-empty algebra over S, and U_1 be a non-empty subalgebra of U_0. Then $\mathrm{Constants}(U_0)$ is a non-empty subset of U_1.

(13) Let S be a non void non empty many sorted signature with constant operations, U_0 be a non-empty algebra over S, and U_1, U_2 be non-empty subalgebras of U_0. Then (the sorts of U_1) $\cap$ (the sorts of U_2) is non-empty.

4. MANY SORTED SUBSETS OF MANY SORTED ALGEBRA

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{SubSorts}(A)$ yielding a set is defined by the condition (Def. 11).

(Def. 11) Let x be a set. Then $x \in \mathrm{SubSorts}(A)$ if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup(\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) x is a subset of U_0, and

(iii) for every subset B of U_0 such that $B = x$ holds B is operations closed and $\mathrm{Constants}(U_0) \subseteq B$ and $A \subseteq B$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. Note that $\mathrm{SubSorts}(A)$ is non empty.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor $\mathrm{SubSorts}(U_0)$ yielding a set is defined by the condition (Def. 12).

(Def. 12) Let x be a set. Then $x \in \mathrm{SubSorts}(U_0)$ if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup(\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) x is a subset of U_0, and

(iii) for every subset B of U_0 such that $B = x$ holds B is operations closed.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. One can verify that $\mathrm{SubSorts}(U_0)$ is non empty.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let e be an element of $\mathrm{SubSorts}(U_0)$. The functor ${}^{@}e$ yields a subset of U_0 and is defined as follows:

(Def. 13) ${}^{@}e = e$.

The following two propositions are true:

(14) For all subsets A, B of U_0 holds $B \in \mathrm{SubSorts}(A)$ iff B is operations closed and $\mathrm{Constants}(U_0) \subseteq B$ and $A \subseteq B$.

(15) For every subset B of U_0 holds $B \in \mathrm{SubSorts}(U_0)$ iff B is operations closed.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let A be a subset of U_0, and let s be a sort symbol of S. The functor $\mathrm{SubSort}(A,s)$ yielding a set is defined as follows:

(Def. 14) For every set x holds $x \in \mathrm{SubSort}(A,s)$ iff there exists a subset B of U_0 such that $B \in \mathrm{SubSorts}(A)$ and $x = B(s)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let A be a subset of U_0, and let s be a sort symbol of S. Note that $\mathrm{SubSort}(A,s)$ is non empty.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{MSSubSort}(A)$ yielding a subset of U_0 is defined as follows:

(Def. 15) For every sort symbol s of S holds $(\mathrm{MSSubSort}(A))(s) = \bigcap \mathrm{SubSort}(A,s)$.

Next we state several propositions:

(16) For every subset A of U_0 holds $\mathrm{Constants}(U_0) \cup A \subseteq \mathrm{MSSubSort}(A)$.

(17) For every subset A of U_0 such that $\mathrm{Constants}(U_0) \cup A$ is non-empty holds $\mathrm{MSSubSort}(A)$ is non-empty.

(18) Let A be a subset of U_0 and B be a subset of U_0. If $B \in \mathrm{SubSorts}(A)$, then $((\mathrm{MSSubSort}(A))^{\#} \cdot \text{the arity of } S)(o) \subseteq (B^{\#} \cdot \text{the arity of } S)(o)$.

(19) Let A be a subset of U_0 and B be a subset of U_0. Suppose $B \in \mathrm{SubSorts}(A)$. Then $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot \text{the arity of } S)(o)) \subseteq (B \cdot \text{the result sort of } S)(o)$.

(20) For every subset A of U_0 holds $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot \text{the arity of } S)(o)) \subseteq (\mathrm{MSSubSort}(A) \cdot \text{the result sort of } S)(o)$.

(21) For every subset A of U_0 holds $\mathrm{MSSubSort}(A)$ is operations closed and $A \subseteq \mathrm{MSSubSort}(A)$.

5. Operations on Many Sorted Algebra and its Subalgebras

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. Let us assume that A is operations closed. The functor $U_0 \restriction A$ yields a strict subalgebra of U_0 and is defined by:

(Def. 16) $U_0 \restriction A = \langle A, (\mathrm{Opers}(U_0, A)$ **qua** many sorted function from $A^{\#} \cdot$ the arity of S into $A \cdot$ the result sort of $S)\rangle$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \cap U_2$ yields a strict subalgebra of U_0 and is defined by the conditions (Def. 17).

(Def. 17)(i) The sorts of $U_1 \cap U_2 =$ (the sorts of U_1) $\cap$ (the sorts of U_2), and

(ii) for every subset B of U_0 such that $B =$ the sorts of $U_1 \cap U_2$ holds B is operations closed and the characteristics of $U_1 \cap U_2 = \mathrm{Opers}(U_0, B)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{Gen}(A)$ yielding a strict subalgebra of U_0 is defined by the conditions (Def. 18).

(Def. 18)(i) A is a subset of $\mathrm{Gen}(A)$, and

(ii) for every subalgebra U_1 of U_0 such that A is a subset of U_1 holds $\mathrm{Gen}(A)$ is a subalgebra of U_1.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, and let A be a non-empty subset of U_0. Observe that $\mathrm{Gen}(A)$ is non-empty.

We now state three propositions:

(22) Let S be a non void non empty many sorted signature, U_0 be a strict algebra over S, and B be a subset of U_0. If $B =$ the sorts of U_0, then $\mathrm{Gen}(B) = U_0$.

(23) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, U_1 be a strict subalgebra of U_0, and B be a subset of U_0. If $B =$ the sorts of U_1, then $\mathrm{Gen}(B) = U_1$.

(24) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, and U_1 be a subalgebra of U_0. Then $\mathrm{Gen}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \sqcup U_2$ yielding a strict subalgebra of U_0 is defined by:

(Def. 19) For every subset A of U_0 such that $A =$ (the sorts of U_1) $\cup$ (the sorts of U_2) holds $U_1 \sqcup U_2 = \mathrm{Gen}(A)$.

Next we state several propositions:

(25) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, U_1 be a subalgebra of U_0, and A, B be subsets of U_0. If $B = A \cup$ the sorts of U_1, then $\mathrm{Gen}(A) \sqcup U_1 = \mathrm{Gen}(B)$.

(26) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, U_1 be a subalgebra of U_0, and B be a subset of U_0. If $B =$ the sorts of U_0, then $\mathrm{Gen}(B) \sqcup U_1 = \mathrm{Gen}(B)$.

(27) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, and U_1, U_2 be subalgebras of U_0. Then $U_1 \sqcup U_2 = U_2 \sqcup U_1$.

(28) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, and U_1, U_2 be strict subalgebras of U_0. Then $U_1 \cap (U_1 \sqcup U_2) = U_1$.

(29) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, and U_1, U_2 be strict subalgebras of U_0. Then $U_1 \cap U_2 \sqcup U_2 = U_2$.

6. LATTICE OF SUBALGEBRAS OF MANY SORTED ALGEBRA

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor Subalgebras(U_0) yielding a set is defined as follows:

(Def. 20) For every x holds $x \in$ Subalgebras(U_0) iff x is a strict subalgebra of U_0.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. Observe that Subalgebras(U_0) is non empty.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The functor MSAlgJoin(U_0) yields a binary operation on Subalgebras(U_0) and is defined as follows:

(Def. 21) For all elements x, y of Subalgebras(U_0) and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $(\mathrm{MSAlgJoin}(U_0))(x, y) = U_1 \sqcup U_2$.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The functor MSAlgMeet(U_0) yields a binary operation on Subalgebras(U_0) and is defined as follows:

(Def. 22) For all elements x, y of Subalgebras(U_0) and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $(\mathrm{MSAlgMeet}(U_0))(x, y) = U_1 \cap U_2$.

In the sequel U_0 denotes a non-empty algebra over S.

The following four propositions are true:

(30) MSAlgJoin(U_0) is commutative.

(31) MSAlgJoin(U_0) is associative.

(32) For every non void non empty many sorted signature S and for every non-empty algebra U_0 over S holds MSAlgMeet(U_0) is commutative.

(33) For every non void non empty many sorted signature S and for every non-empty algebra U_0 over S holds MSAlgMeet(U_0) is associative.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The lattice of subalgebras of U_0 yields a strict lattice and is defined by:

(Def. 23) The lattice of subalgebras of $U_0 = \langle \mathrm{Subalgebras}(U_0), \mathrm{MSAlgJoin}(U_0), \mathrm{MSAlgMeet}(U_0) \rangle$.

The following proposition is true

(34) Let S be a non void non empty many sorted signature and U_0 be a non-empty algebra over S. Then the lattice of subalgebras of U_0 is bounded.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. Observe that the lattice of subalgebras of U_0 is bounded.

One can prove the following propositions:

(35) Let S be a non void non empty many sorted signature and U_0 be a non-empty algebra over S. Then $\perp_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

(36) Let S be a non void non empty many sorted signature, U_0 be a non-empty algebra over S, and B be a subset of U_0. If $B =$ the sorts of U_0, then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(B)$.

(37) Let S be a non void non empty many sorted signature and U_0 be a strict non-empty algebra over S. Then $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

(38) Let S be a non void non empty many sorted signature and U_0 be an algebra over S. Then $\langle$the sorts of U_0, the characteristics of $U_0\rangle$ is a subalgebra of U_0.

(39) Let S be a non void non empty many sorted signature and U_0 be a non-empty algebra over S. Then $\langle$the sorts of U_0, the characteristics of $U_0\rangle$ is non-empty.

(40) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, and A be a subset of U_0. Then the sorts of $U_0 \in \mathrm{SubSorts}(A)$.

(41) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, and A be a subset of U_0. Then $\mathrm{SubSorts}(A) \subseteq \mathrm{SubSorts}(U_0)$.

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