JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Many Sorted Algebras

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Summary. The basic purpose of the paper is to prepare preliminaries of the theory of many sorted algebras. The concept of the signature of a many sorted algebra is introduced as well as the concept of many sorted algebra itself. Some auxiliary related notions are defined. The correspondence between (1 sorted) universal algebras [8] and many sorted algebras with one sort only is described by introducing two functors mapping one into the other. The construction is done this way that the composition of both functors is the identity on universal algebras.

MML Identifier: `MSUALG_1`.

WWW: `http://mizar.org/JFM/Vol6/msualg_1.html`

The articles [10], [13], [12], [14], [4], [5], [2], [9], [6], [7], [1], [11], [3], and [8] provide the notation and terminology for this paper.

1. Preliminaries

In this paper i, j denote sets and I denotes a set.

One can prove the following proposition

(1) It is not true that there exists a non-empty many sorted set M indexed by I such that $\emptyset \in \operatorname{rng} M$.

In this article we present several logical schemes. The scheme *MSSEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a many sorted set f indexed by $\mathcal{A}$ such that for every i such that $i \in \mathcal{A}$ holds $\mathcal{P}[i, f(i)]$

provided the following requirement is met:

- For every i such that $i \in \mathcal{A}$ there exists j such that $\mathcal{P}[i, j]$.

The scheme *MSSLambda* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a many sorted set f indexed by $\mathcal{A}$ such that for every i such that $i \in \mathcal{A}$ holds $f(i) = \mathcal{F}(i)$

for all values of the parameters.

Let I be a set and let M be a many sorted set indexed by I. A component of M is an element of $\operatorname{rng} M$.

One can prove the following propositions:

(2) Let I be a non empty set, M be a many sorted set indexed by I, and A be a component of M. Then there exists i such that $i \in I$ and $A = M(i)$.

(3) For every many sorted set M indexed by I and for every i such that $i \in I$ holds $M(i)$ is a component of M.

Let us consider I and let B be a many sorted set indexed by I. A many sorted set indexed by I is said to be an element of B if:

(Def. 1) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is an element of $B(i)$.

2. Auxiliary functors

Let us consider I, let A be a many sorted set indexed by I, and let B be a many sorted set indexed by I. A many sorted set indexed by I is said to be a many sorted function from A into B if:

(Def. 2) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider I, let A be a many sorted set indexed by I, and let B be a many sorted set indexed by I. Note that every many sorted function from A into B is function yielding.

Let I be a set and let M be a many sorted set indexed by I. The functor $M^{\#}$ yields a many sorted set indexed by I^* and is defined by:

(Def. 3) For every element i of I^* holds $M^{\#}(i) = \prod(M \cdot i)$.

Let I be a set and let M be a non-empty many sorted set indexed by I. One can check that $M^{\#}$ is non-empty.

Let us consider I, let J be a non empty set, let O be a function from I into J, and let F be a many sorted set indexed by J. Then $F \cdot O$ is a many sorted set indexed by I.

Let us consider I, let J be a non empty set, let O be a function from I into J, and let F be a non-empty many sorted set indexed by J. Then $F \cdot O$ is a non-empty many sorted set indexed by I.

Let a be a set. The functor $\square \longmapsto a$ yielding a function from $\mathbb{N}$ into $\{a\}^*$ is defined as follows:

(Def. 4) For every natural number n holds $(\square \longmapsto a)(n) = n \mapsto a$.

In the sequel D denotes a non empty set and n denotes a natural number.

We now state two propositions:

(4) For all sets a, b holds $(\{a\} \longmapsto b) \cdot (n \mapsto a) = n \mapsto b$.

(5) For every set a and for every many sorted set M indexed by $\{a\}$ such that $M = \{a\} \longmapsto D$ holds $(M^{\#} \cdot (\square \longmapsto a))(n) = D^{\mathrm{Seg}\, n}$.

Let us consider I, i. Then $I \longmapsto i$ is a function from I into $\{i\}$.

3. Many sorted signatures

We introduce many sorted signatures which are extensions of 1-sorted structure and are systems

⟨ a carrier, operation symbols, an arity, a result sort ⟩,

where the carrier is a set, the operation symbols constitute a set, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

Let I_1 be a many sorted signature. We say that I_1 is void if and only if:

(Def. 5) The operation symbols of $I_1 = \emptyset$.

Let us observe that there exists a many sorted signature which is void, strict, and non empty and there exists a many sorted signature which is non void, strict, and non empty.

In the sequel S is a non empty many sorted signature.

Let us consider S. A sort symbol of S is an element of S. An operation symbol of S is an element of the operation symbols of S.

Let S be a non void non empty many sorted signature and let o be an operation symbol of S. The functor $\mathrm{Arity}(o)$ yields an element of (the carrier of $S)^*$ and is defined by:

(Def. 6) $\text{Arity}(o) = (\text{the arity of } S)(o)$.

The result sort of o yielding an element of S is defined by:

(Def. 7) The result sort of $o = (\text{the result sort of } S)(o)$.

4. MANY SORTED ALGEBRAS

Let S be a 1-sorted structure. We introduce many-sorted structures over S which are systems
⟨ sorts ⟩,
where the sorts constitute a many sorted set indexed by the carrier of S.

Let us consider S. We introduce algebras over S which are extensions of many-sorted structure over S and are systems
⟨ sorts, a characteristics ⟩,
where the sorts constitute a many sorted set indexed by the carrier of S and the characteristics is a many sorted function from the $\text{sorts}^{\#} \cdot$ the arity of S into the sorts $\cdot$the result sort of S.

Let S be a 1-sorted structure and let A be a many-sorted structure over S. We say that A is non-empty if and only if:

(Def. 8) The sorts of A are non-empty.

Let us consider S. Observe that there exists an algebra over S which is strict and non-empty.

Let S be a 1-sorted structure. One can verify that there exists a many-sorted structure over S which is strict and non-empty.

Let S be a 1-sorted structure and let A be a non-empty many-sorted structure over S. Note that the sorts of A is non-empty.

Let us consider S and let A be a non-empty algebra over S. Observe that every component of the sorts of A is non empty and every component of $(\text{the sorts of } A)^{\#}$ is non empty.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be an algebra over S. The functor $\text{Args}(o,A)$ yields a component of $(\text{the sorts of } A)^{\#}$ and is defined as follows:

(Def. 9) $\text{Args}(o,A) = ((\text{the sorts of } A)^{\#} \cdot \text{the arity of } S)(o)$.

The functor $\text{Result}(o,A)$ yielding a component of the sorts of A is defined by:

(Def. 10) $\text{Result}(o,A) = ((\text{the sorts of } A) \cdot (\text{the result sort of } S))(o)$.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be an algebra over S. The functor $\text{Den}(o,A)$ yields a function from $\text{Args}(o,A)$ into $\text{Result}(o,A)$ and is defined as follows:

(Def. 11) $\text{Den}(o,A) = (\text{the characteristics of } A)(o)$.

We now state the proposition

(6) Let S be a non void non empty many sorted signature, o be an operation symbol of S, and A be a non-empty algebra over S. Then $\text{Den}(o,A)$ is non empty.

5. UNIVERSAL ALGEBRAS AS MANY SORTED

The following propositions are true:

(7) Let C be a set, A, B be non empty sets, F be a partial function from C to A, and G be a function from A into B. Then $G \cdot F$ is a function from $\text{dom} F$ into B.

(8) For every homogeneous quasi total non empty partial function h from D^* to D holds $\text{dom}\, h = D^{\text{Seg arity}\, h}$.

(9) For every universal algebra A holds signature$A$ is non empty.

6. UNIVERSAL ALGEBRAS FOR MANY SORTED ALGEBRAS WITH ONE SORT

Let A be a universal algebra. Then signature$A$ is a finite sequence of elements of $\mathbb{N}$.

Let I_1 be a many sorted signature. We say that I_1 is segmental if and only if:

(Def. 12) There exists n such that the operation symbols of $I_1 = \mathrm{Seg}\, n$.

We now state the proposition

(10) Let S be a non empty many sorted signature. Suppose S is trivial. Let A be an algebra over S and c_1, c_2 be components of the sorts of A. Then $c_1 = c_2$.

Let us note that there exists a many sorted signature which is segmental, trivial, non void, strict, and non empty.

Let A be a universal algebra. The functor $\mathrm{MSSign}(A)$ yielding a non void strict segmental trivial many sorted signature is defined by the conditions (Def. 13).

(Def. 13)(i) The carrier of $\mathrm{MSSign}(A) = \{0\}$,

(ii) the operation symbols of $\mathrm{MSSign}(A) = \mathrm{dom}\,\mathrm{signature}A$,

(iii) the arity of $\mathrm{MSSign}(A) = (\square \longmapsto 0) \cdot \mathrm{signature}A$, and

(iv) the result sort of $\mathrm{MSSign}(A) = \mathrm{dom}\,\mathrm{signature}A \longmapsto 0$.

Let A be a universal algebra. Note that $\mathrm{MSSign}(A)$ is non empty.

Let A be a universal algebra. The functor $\mathrm{MSSorts}(A)$ yielding a non-empty many sorted set indexed by the carrier of $\mathrm{MSSign}(A)$ is defined by:

(Def. 14) $\mathrm{MSSorts}(A) = \{0\} \longmapsto$ the carrier of A.

Let A be a universal algebra. The functor $\mathrm{MSCharact}(A)$ yielding a many sorted function from $(\mathrm{MSSorts}(A))^{\#} \cdot$ the arity of $\mathrm{MSSign}(A)$ into $\mathrm{MSSorts}(A) \cdot$ the result sort of $\mathrm{MSSign}(A)$ is defined as follows:

(Def. 15) $\mathrm{MSCharact}(A) =$ the characteristic of A.

Let A be a universal algebra. The functor $\mathrm{MSAlg}(A)$ yields a strict algebra over $\mathrm{MSSign}(A)$ and is defined by:

(Def. 16) $\mathrm{MSAlg}(A) = \langle \mathrm{MSSorts}(A), \mathrm{MSCharact}(A) \rangle$.

Let A be a universal algebra. One can verify that $\mathrm{MSAlg}(A)$ is non-empty.

Let M_1 be a trivial non empty many sorted signature and let A be an algebra over M_1. The sort of A yields a set and is defined as follows:

(Def. 17) There exists a component c of the sorts of A such that the sort of $A = c$.

Let M_1 be a trivial non empty many sorted signature and let A be a non-empty algebra over M_1. Observe that the sort of A is non empty.

We now state four propositions:

(11) Let M_1 be a segmental trivial non void non empty many sorted signature, i be an operation symbol of M_1, and A be a non-empty algebra over M_1. Then $\mathrm{Args}(i, A) =$ (the sort of $A)^{\mathrm{len}\,\mathrm{Arity}(i)}$.

(12) For every non empty set A and for every n holds $A^n \subseteq A^*$.

(13) Let M_1 be a segmental trivial non void non empty many sorted signature, i be an operation symbol of M_1, and A be a non-empty algebra over M_1. Then $\mathrm{Args}(i, A) \subseteq$ (the sort of $A)^*$.

(14) Let M_1 be a segmental trivial non void non empty many sorted signature and A be a non-empty algebra over M_1. Then the characteristics of A is a finite sequence of elements of (the sort of $A)^* \dot{\rightarrow}$the sort of A.

Let M_1 be a segmental trivial non void non empty many sorted signature and let A be a non-empty algebra over M_1. The functor $\mathrm{charact}(A)$ yielding a finite sequence of operational functions of the sort of A is defined as follows:

(Def. 18) $\mathrm{charact}(A) =$ the characteristics of A.

In the sequel M_1 is a segmental trivial non void non empty many sorted signature and A is a non-empty algebra over M_1.

Let us consider M_1, A. The functor $\mathrm{Alg}_{\mathbf{1}}(A)$ yields a non-empty strict universal algebra and is defined by:

(Def. 19) $\mathrm{Alg}_{\mathbf{1}}(A) = \langle$the sort of A, $\mathrm{charact}(A)\rangle$.

Next we state two propositions:

(15) For every strict universal algebra A holds $A = \mathrm{Alg}_{\mathbf{1}}(\mathrm{MSAlg}(A))$.

(16) Let A be a universal algebra and f be a function from $\mathrm{dom}\,\mathrm{signature}A$ into $\{0\}^*$. If $f = (\square \longmapsto 0) \cdot \mathrm{signature}A$, then $\mathrm{MSSign}(A) = \langle \{0\}, \mathrm{dom}\,\mathrm{signature}A, f, \mathrm{dom}\,\mathrm{signature}A \longmapsto 0\rangle$.

ACKNOWLEDGMENTS

I would like to express my gratitude to Czesław Byliński, whose remarks enabled me to enhance the paper.

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Received April 21, 1994

Published January 2, 2004