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The Correspondence Between Monotonic Many Sorted Signatures and Well-Founded Graphs. Part I[1]

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Summary. We prove a number of auxiliary facts about graphs, mainly about vertex sequences of chains and oriented chains. Then we define a graph to be *well-founded* if for each vertex in the graph the length of oriented chains ending at the vertex is bounded. A *well-founded* graph does not have directed cycles or infinite descending chains. In the second part of the article we prove some auxiliary facts about free algebras and locally-finite algebras.

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The articles [27], [17], [30], [2], [1], [25], [4], [31], [14], [16], [15], [19], [11], [21], [23], [20], [3], [6], [7], [5], [8], [18], [12], [28], [29], [13], [22], [26], [24], [9], and [10] provide the notation and terminology for this paper.

1. Some properties of graphs

One can prove the following proposition

(1) For every finite function f such that for every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is finite holds $\prod f$ is finite.

In the sequel G denotes a graph and m, n denote natural numbers.

Let G be a graph. Let us note that the chain of G can be characterized by the following (equivalent) condition:

(Def. 1) It is a finite sequence of elements of the edges of G and there exists a finite sequence of elements of the vertices of G which is vertex sequence of it.

The following proposition is true

(2) For all finite sequences p, q such that $n \leq \operatorname{len} p$ holds $\langle p(1), \ldots, p(n) \rangle = \langle (p \frown q)(1), \ldots, (p \frown q)(n) \rangle$.

Let G be a graph and let I_1 be a chain of G. We introduce I_1 is directed as a synonym of I_1 is oriented.

Let G be a graph and let I_1 be a chain of G. We say that I_1 is cyclic if and only if:

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(Def. 2) There exists a finite sequence p of elements of the vertices of G such that p is vertex sequence of I_1 and $p(1) = p(\operatorname{len} p)$.

Let I_1 be a graph. We say that I_1 is empty if and only if:

(Def. 3) The edges of I_1 are empty.

Let us note that there exists a graph which is empty.
One can prove the following proposition

(3) For every graph G holds $\operatorname{rng}(\text{the source of } G) \cup \operatorname{rng}(\text{the target of } G) \subseteq$ the vertices of G.

Let us note that there exists a graph which is finite, simple, connected, non empty, and strict.
Let G be a non empty graph. Note that the edges of G is non empty.
We now state two propositions:

(4) Let e be a set and s, t be elements of the vertices of G. Suppose $s = (\text{the source of } G)(e)$ and $t = (\text{the target of } G)(e)$. Then $\langle s, t \rangle$ is vertex sequence of $\langle e \rangle$.

(5) For every set e such that $e \in$ the edges of G holds $\langle e \rangle$ is a directed chain of G.

In the sequel G denotes a non empty graph.
Let us consider G. Note that there exists a chain of G which is directed, non empty, and one-to-one.
The following propositions are true:

(6) Let G be a graph, c be a chain of G, and v_1 be a finite sequence of elements of the vertices of G. If c is cyclic and v_1 is vertex sequence of c, then $v_1(1) = v_1(\operatorname{len} v_1)$.

(7) Let G be a graph and e be a set. Suppose $e \in$ the edges of G. Let f_1 be a directed chain of G. If $f_1 = \langle e \rangle$, then $\text{vertex-seq}(f_1) = \langle (\text{the source of } G)(e), (\text{the target of } G)(e) \rangle$.

(8) For every finite sequence f holds $\operatorname{len}\langle f(m), \ldots, f(n) \rangle \leq \operatorname{len} f$.

(9) For every directed chain c of G such that $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ holds $\langle c(m), \ldots, c(n) \rangle$ is a directed chain of G.

(10) For every non empty directed chain o_1 of G holds $\operatorname{len} \text{vertex-seq}(o_1) = \operatorname{len} o_1 + 1$.

Let us consider G and let o_1 be a directed non empty chain of G. Note that $\text{vertex-seq}(o_1)$ is non empty.
We now state several propositions:

(11) Let o_1 be a directed non empty chain of G and given n. Suppose $1 \leq n$ and $n \leq \operatorname{len} o_1$. Then $(\text{vertex-seq}(o_1))(n) = (\text{the source of } G)(o_1(n))$ and $(\text{vertex-seq}(o_1))(n+1) = (\text{the target of } G)(o_1(n))$.

(12) For every non empty finite sequence f such that $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} f$ holds $\langle f(m), \ldots, f(n) \rangle$ is non empty.

(13) For all directed chains c, c_1 of G such that $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ holds $\text{vertex-seq}(c_1) = \langle (\text{vertex-seq}(c))(m), \ldots, (\text{vertex-seq}(c))(n+1) \rangle$.

(14) For every directed non empty chain o_1 of G holds $(\text{vertex-seq}(o_1))(\operatorname{len} o_1 + 1) = (\text{the target of } G)(o_1(\operatorname{len} o_1))$.

(15) For all directed non empty chains c_1, c_2 of G holds $(\text{vertex-seq}(c_1))(\operatorname{len} c_1 + 1) = (\text{vertex-seq}(c_2))(1)$ iff $c_1 \frown c_2$ is a directed non empty chain of G.

(16) For all directed non empty chains c, c_1, c_2 of G such that $c = c_1 \frown c_2$ holds $(\text{vertex-seq}(c))(1) = (\text{vertex-seq}(c_1))(1)$ and $(\text{vertex-seq}(c))(\operatorname{len} c + 1) = (\text{vertex-seq}(c_2))(\operatorname{len} c_2 + 1)$.

(17) For every directed non empty chain o_1 of G such that o_1 is cyclic holds $(\text{vertex-seq}(o_1))(1) = (\text{vertex-seq}(o_1))(\operatorname{len} o_1 + 1)$.

(18) Let c be a directed non empty chain of G. Suppose c is cyclic. Let given n. Then there exists a directed chain c_3 of G such that $\operatorname{len} c_3 = n$ and $c_3 \frown c$ is a directed non empty chain of G.

Let I_1 be a graph. We say that I_1 is directed cycle-less if and only if:

(Def. 4) For every directed chain d_1 of I_1 such that d_1 is non empty holds d_1 is non cyclic.

We introduce I_1 has directed cycle as an antonym of I_1 is directed cycle-less.

One can verify that every graph which is empty is also directed cycle-less.

Let I_1 be a graph. We say that I_1 is well-founded if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let v be an element of the vertices of I_1. Then there exists n such that for every directed chain c of I_1 if c is non empty and $(\text{vertex-seq}(c))(\operatorname{len} c + 1) = v$, then $\operatorname{len} c \leq n$.

Let G be an empty graph. Note that every chain of G is empty.

Let us note that every graph which is empty is also well-founded.

Let us observe that every graph which is non well-founded is also non empty.

One can verify that there exists a graph which is well-founded.

Let us note that every graph which is well-founded is also directed cycle-less.

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Let us mention that there exists a graph which is directed cycle-less.

Next we state the proposition

(19) For every decorated tree t and for every node p of t and for every natural number k holds $p \restriction k$ is a node of t.

2. Some properties of many sorted algebras

One can prove the following two propositions:

(20) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Suppose t is not root. Then there exists an operation symbol o of S such that $t(\emptyset) = \langle o, \text{the carrier of } S \rangle$.

(21) Let S be a non void non empty many sorted signature, A be an algebra over S, G be a generator set of A, and B be a subset of A. If $G \subseteq B$, then B is a generator set of A.

Let S be a non void non empty many sorted signature and let A be a finitely-generated non-empty algebra over S. Note that there exists a generator set of A which is non-empty and locally-finite.

One can prove the following propositions:

(22) Let S be a non void non empty many sorted signature, A be a non-empty algebra over S, and X be a non-empty generator set of A. Then there exists a many sorted function from $\text{Free}(X)$ into A which is an epimorphism of $\text{Free}(X)$ onto A.

(23) Let S be a non void non empty many sorted signature, A be a non-empty algebra over S, and X be a non-empty generator set of A. If A is non locally-finite, then $\text{Free}(X)$ is non locally-finite.

Let S be a non void non empty many sorted signature, let X be a non-empty locally-finite many sorted set indexed by the carrier of S, and let v be a sort symbol of S. Note that $\text{FreeGenerator}(v, X)$ is finite.

We now state the proposition

(25)[1] Let S be a non void non empty many sorted signature, A be a non-empty algebra over S, and o be an operation symbol of S. If (the arity of S)$(o) = \emptyset$, then $\mathrm{dom}\,\mathrm{Den}(o, A) = \{\emptyset\}$.

Let I_1 be a non void non empty many sorted signature. We say that I_1 is finitely operated if and only if:

(Def. 6) For every sort symbol s of I_1 holds $\{o; o$ ranges over operation symbols of I_1: the result sort of $o = s\}$ is finite.

We now state three propositions:

(26) Let S be a non void non empty many sorted signature, A be a non-empty algebra over S, and v be a sort symbol of S. If S is finitely operated, then $\mathrm{Constants}(A, v)$ is finite.

(27) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, and v be a sort symbol of S. Then $\{t; t$ ranges over elements of (the sorts of $\mathrm{Free}(X))(v)$: $\mathrm{depth}(t) = 0\} = \mathrm{FreeGenerator}(v, X) \cup \mathrm{Constants}(\mathrm{Free}(X), v)$.

(28) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, v, v_2 be sort symbols of S, o be an operation symbol of S, t be an element of (the sorts of $\mathrm{Free}(X))(v)$, a be an argument sequence of $\mathrm{Sym}(o, X)$, k be a natural number, and a_1 be an element of (the sorts of $\mathrm{Free}(X))(v_2)$. If $t = \langle o$, the carrier of $S\rangle$-tree(a) and $k \in \mathrm{dom}\,a$ and $a_1 = a(k)$, then $\mathrm{depth}(a_1) < \mathrm{depth}(t)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/trees_1.html`.

[4] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[5] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/funct_6.html`.

[6] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/trees_2.html`.

[7] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/trees_3.html`.

[8] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/trees_4.html`.

[9] Grzegorz Bancerek. Subtrees. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/trees_9.html`.

[10] Grzegorz Bancerek. Terms over many sorted universal algebra. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msaterm.html`.

[11] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[12] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/dtconstr.html`.

[13] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_2.html`.

[14] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[15] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[16] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[17] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[1] The proposition (24) has been removed.

[18] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lang1.html.

[19] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[20] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_T$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[21] Krzysztof Hryniewiecki. Graphs. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/graph_1.html.

[22] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[23] Yatsuka Nakamura and Piotr Rudnicki. Vertex sequences induced by chains. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/graph_2.html.

[24] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree2.html.

[25] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/prob_1.html.

[26] Beata Perkowska. Free many sorted universal algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree.html.

[27] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[28] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[29] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[30] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[31] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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