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Terms Over Many Sorted Universal Algebra[1]

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Summary. Pure terms (without constants) over a signature of many sorted universal algebra and terms with constants from algebra are introduced. Facts on evaluation of a term in some valuation are proved.

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The articles [18], [12], [23], [21], [24], [10], [8], [11], [14], [1], [4], [16], [2], [22], [3], [13], [5], [6], [7], [19], [20], [9], [15], and [17] provide the notation and terminology for this paper.

1. Terms over a Signature and over an Algebra

Let I be a non empty set, let X be a non-empty many sorted set indexed by I, and let i be an element of I. Observe that $X(i)$ is non empty.

In the sequel S denotes a non void non empty many sorted signature and V denotes a non-empty many sorted set indexed by the carrier of S.

Let us consider S and let V be a many sorted set indexed by the carrier of S. The functor S-Terms(V) yielding a subset of FinTrees(the carrier of DTConMSA(V)) is defined by:

(Def. 1) S-Terms$(V) = \mathrm{TS}(\mathrm{DTConMSA}(V))$.

Let us consider S, V. Note that S-Terms(V) is non empty.

Let us consider S, V. A term of S over V is an element of S-Terms(V).

In the sequel A is an algebra over S and t is a term of S over V.

Let us consider S, V and let o be an operation symbol of S. Then Sym(o,V) is a nonterminal of DTConMSA(V).

Let us consider S, V and let s_1 be a nonterminal of DTConMSA(V). A finite sequence of elements of S-Terms(V) is said to be an argument sequence of s_1 if:

(Def. 2) It is a subtree sequence joinable by s_1.

We now state the proposition

(1) Let o be an operation symbol of S and a be a finite sequence. Then $\langle o,$ the carrier of $S\rangle$-tree$(a) \in S$-Terms(V) and a is decorated tree yielding if and only if a is an argument sequence of Sym(o,V).

The scheme *TermInd* deals with a non void non empty many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

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For every term t of $\mathcal{A}$ over $\mathcal{B}$ holds $\mathcal{P}[t]$

provided the following conditions are satisfied:

- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{B}(s)$ holds $\mathcal{P}$[the root tree of $\langle v, s\rangle$], and
- Let o be an operation symbol of $\mathcal{A}$ and p be an argument sequence of $\mathrm{Sym}(o, \mathcal{B})$. Suppose that for every term t of $\mathcal{A}$ over $\mathcal{B}$ such that $t \in \operatorname{rng} p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\langle o,$ the carrier of $\mathcal{A}\rangle$-tree$(p)]$.

Let us consider S, A, V. A term of A over V is a term of S over (the sorts of A) $\cup V$.

Let us consider S, A, V and let o be an operation symbol of S. An argument sequence of o, A, and V is an argument sequence of $\mathrm{Sym}(o,$(the sorts of $A) \cup V)$.

The scheme *CTermInd* deals with a non void non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{B}$ over $\mathcal{C}$ holds $\mathcal{P}[t]$

provided the parameters meet the following requirements:

- For every sort symbol s of $\mathcal{A}$ and for every element x of (the sorts of $\mathcal{B})(s)$ holds $\mathcal{P}$[the root tree of $\langle x, s\rangle$],
- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{C}(s)$ holds $\mathcal{P}$[the root tree of $\langle v, s\rangle$], and
- Let o be an operation symbol of $\mathcal{A}$ and p be an argument sequence of o, $\mathcal{B}$, and $\mathcal{C}$. Suppose that for every term t of $\mathcal{B}$ over $\mathcal{C}$ such that $t \in \operatorname{rng} p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\mathrm{Sym}(o,$(the sorts of $\mathcal{B}) \cup \mathcal{C})$-tree$(p)]$.

Let us consider S, V, t and let p be a node of t. Then $t(p)$ is a symbol of DTConMSA(V).

Let us consider S, V. Note that every term of S over V is finite.

The following propositions are true:

(2)(i) There exists a sort symbol s of S and there exists an element v of $V(s)$ such that $t(\emptyset) = \langle v, s\rangle$, or

(ii) $t(\emptyset) \in [:$the operation symbols of S, $\{$the carrier of $S\}:]$.

(3) Let t be a term of A over V. Then

(i) there exists a sort symbol s of S and there exists a set x such that $x \in$ (the sorts of $A)(s)$ and $t(\emptyset) = \langle x, s\rangle$, or

(ii) there exists a sort symbol s of S and there exists an element v of $V(s)$ such that $t(\emptyset) = \langle v, s\rangle$, or

(iii) $t(\emptyset) \in [:$the operation symbols of S, $\{$the carrier of $S\}:]$.

(4) For every sort symbol s of S and for every element v of $V(s)$ holds the root tree of $\langle v, s\rangle$ is a term of S over V.

(5) For every sort symbol s of S and for every element v of $V(s)$ such that $t(\emptyset) = \langle v, s\rangle$ holds $t =$ the root tree of $\langle v, s\rangle$.

(6) Let s be a sort symbol of S and x be a set. Suppose $x \in$ (the sorts of $A)(s)$. Then the root tree of $\langle x, s\rangle$ is a term of A over V.

(7) Let t be a term of A over V, s be a sort symbol of S, and x be a set. If $x \in$ (the sorts of $A)(s)$ and $t(\emptyset) = \langle x, s\rangle$, then $t =$ the root tree of $\langle x, s\rangle$.

(8) For every sort symbol s of S and for every element v of $V(s)$ holds the root tree of $\langle v, s\rangle$ is a term of A over V.

(9) Let t be a term of A over V, s be a sort symbol of S, and v be an element of $V(s)$. If $t(\emptyset) = \langle v, s\rangle$, then $t =$ the root tree of $\langle v, s\rangle$.

(10) Let o be an operation symbol of S. Suppose $t(\emptyset) = \langle o,$ the carrier of $S\rangle$. Then there exists an argument sequence a of $\mathrm{Sym}(o, V)$ such that $t = \langle o,$ the carrier of $S\rangle$-tree(a).

Let us consider S, let A be a non-empty algebra over S, let us consider V, let s be a sort symbol of S, and let x be an element of (the sorts of A)(s). The functor $x_{A,V}$ yields a term of A over V and is defined as follows:

(Def. 3) $x_{A,V} =$ the root tree of $\langle x, s\rangle$.

Let us consider S, A, V, let s be a sort symbol of S, and let v be an element of $V(s)$. The functor v_A yields a term of A over V and is defined by:

(Def. 4) $v_A =$ the root tree of $\langle v, s\rangle$.

Let us consider S, V, let s_1 be a nonterminal of DTConMSA(V), and let p be an argument sequence of s_1. Then s_1-tree(p) is a term of S over V.

The scheme *TermInd2* deals with a non void non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{B}$ over $\mathcal{C}$ holds $\mathcal{P}[t]$

provided the parameters meet the following conditions:

- For every sort symbol s of $\mathcal{A}$ and for every element x of (the sorts of $\mathcal{B}$)(s) holds $\mathcal{P}[x_{\mathcal{B},\mathcal{C}}]$,
- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{C}(s)$ holds $\mathcal{P}[v_{\mathcal{B}}]$, and
- Let o be an operation symbol of $\mathcal{A}$ and p be an argument sequence of Sym$(o,$(the sorts of $\mathcal{B}) \cup \mathcal{C})$. Suppose that for every term t of $\mathcal{B}$ over $\mathcal{C}$ such that $t \in \operatorname{rng} p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[$Sym$(o,$(the sorts of $\mathcal{B}) \cup \mathcal{C})$-tree$(p)]$.

2. Sort of a Term

We now state three propositions:

(11) For every term t of S over V there exists a sort symbol s of S such that $t \in$ FreeSort(V, s).

(12) For every sort symbol s of S holds FreeSort$(V, s) \subseteq S$-Terms(V).

(13) S-Terms$(V) = \bigcup$FreeSorts(V).

Let us consider S, V, t. The sort of t yields a sort symbol of S and is defined by:

(Def. 5) $t \in$ FreeSort$(V,$ the sort of $t)$.

The following propositions are true:

(14) Let s be a sort symbol of S and v be an element of $V(s)$. If $t =$ the root tree of $\langle v, s\rangle$, then the sort of $t = s$.

(15) Let t be a term of A over V, s be a sort symbol of S, and x be a set. Suppose $x \in$ (the sorts of A)(s) and $t =$ the root tree of $\langle x, s\rangle$. Then the sort of $t = s$.

(16) Let t be a term of A over V, s be a sort symbol of S, and v be an element of $V(s)$. If $t =$ the root tree of $\langle v, s\rangle$, then the sort of $t = s$.

(17) Let o be an operation symbol of S. Suppose $t(\emptyset) = \langle o,$ the carrier of $S\rangle$. Then the sort of $t =$ the result sort of o.

(18) Let A be a non-empty algebra over S, s be a sort symbol of S, and x be an element of (the sorts of A)(s). Then the sort of $x_{A,V} = s$.

(19) For every sort symbol s of S and for every element v of $V(s)$ holds the sort of $v_A = s$.

(20) Let o be an operation symbol of S and p be an argument sequence of Sym(o,V). Then the sort of (Sym(o,V)-tree(p) **qua** term of S over V) $=$ the result sort of o.

3. Argument Sequence

Next we state several propositions:

(21) Let o be an operation symbol of S and a be a finite sequence of elements of S-Terms(V). Then a is an argument sequence of $\mathrm{Sym}(o,V)$ if and only if $\mathrm{Sym}(o,V) \Rightarrow$ the roots of a.

(22) Let o be an operation symbol of S and a be an argument sequence of $\mathrm{Sym}(o,V)$. Then $\mathrm{len}\, a = \mathrm{len}\,\mathrm{Arity}(o)$ and $\mathrm{dom}\, a = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number i such that $i \in \mathrm{dom}\, a$ holds $a(i)$ is a term of S over V.

(23) Let o be an operation symbol of S, a be an argument sequence of $\mathrm{Sym}(o,V)$, and i be a natural number. Suppose $i \in \mathrm{dom}\, a$. Let t be a term of S over V. Suppose $t = a(i)$. Then

(i) $t = (a$ **qua** finite sequence of elements of S-Terms(V) **qua** non empty set$)_i$,

(ii) the sort of $t = \mathrm{Arity}(o)(i)$, and

(iii) the sort of $t = \mathrm{Arity}(o)_i$.

(24) Let o be an operation symbol of S and a be a finite sequence. Suppose that

(i) $\mathrm{len}\, a = \mathrm{len}\,\mathrm{Arity}(o)$ or $\mathrm{dom}\, a = \mathrm{dom}\,\mathrm{Arity}(o)$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, a$ there exists a term t of S over V such that $t = a(i)$ and the sort of $t = \mathrm{Arity}(o)(i)$ or for every natural number i such that $i \in \mathrm{dom}\, a$ there exists a term t of S over V such that $t = a(i)$ and the sort of $t = \mathrm{Arity}(o)_i$.

Then a is an argument sequence of $\mathrm{Sym}(o,V)$.

(25) Let o be an operation symbol of S and a be a finite sequence of elements of S-Terms(V). Suppose that

(i) $\mathrm{len}\, a = \mathrm{len}\,\mathrm{Arity}(o)$ or $\mathrm{dom}\, a = \mathrm{dom}\,\mathrm{Arity}(o)$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, a$ and for every term t of S over V such that $t = a(i)$ holds the sort of $t = \mathrm{Arity}(o)(i)$ or for every natural number i such that $i \in \mathrm{dom}\, a$ and for every term t of S over V such that $t = a(i)$ holds the sort of $t = \mathrm{Arity}(o)_i$.

Then a is an argument sequence of $\mathrm{Sym}(o,V)$.

(26) Let S be a non void non empty many sorted signature and V_1, V_2 be non-empty many sorted sets indexed by the carrier of S. If $V_1 \subseteq V_2$, then every term of S over V_1 is a term of S over V_2.

(27) Let S be a non void non empty many sorted signature, A be an algebra over S, and V be a non-empty many sorted set indexed by the carrier of S. Then every term of S over V is a term of A over V.

4. Compound Terms

Let S be a non void non empty many sorted signature and let V be a non-empty many sorted set indexed by the carrier of S. A term of S over V is called a compound term of S over V if:

(Def. 6) $\mathrm{It}(\emptyset) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!]$.

Let S be a non void non empty many sorted signature and let V be a non-empty many sorted set indexed by the carrier of S. A non empty subset of S-Terms(V) is said to be a set with a compound term of S over V if:

(Def. 7) There exists a compound term t of S over V such that $t \in$ it.

The following propositions are true:

(28) If t is not root, then t is a compound term of S over V.

(29) For every node p of t holds $t{\restriction}p$ is a term of S over V.

Let S be a non void non empty many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let t be a term of S over V, and let p be a node of t. Then $t{\restriction}p$ is a term of S over V.

5. Evaluation of Terms

Let S be a non void non empty many sorted signature and let A be an algebra over S. A non-empty many sorted set indexed by the carrier of S is said to be a variables family of A if:

(Def. 8) It misses the sorts of A.

We now state the proposition

(30) Let V be a variables family of A, s be a sort symbol of S, and x be a set. If $x \in$ (the sorts of $A)(s)$, then for every element v of $V(s)$ holds $x \neq v$.

Let S be a non void non empty many sorted signature, let A be a non-empty algebra over S, let V be a non-empty many sorted set indexed by the carrier of S, let t be a term of A over V, let f be a many sorted function from V into the sorts of A, and let v_1 be a finite decorated tree. We say that v_1 is an evaluation of t w.r.t. f if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $\operatorname{dom} v_1 = \operatorname{dom} t$, and

(ii) for every node p of v_1 holds for every sort symbol s of S and for every element v of $V(s)$ such that $t(p) = \langle v, s\rangle$ holds $v_1(p) = f(s)(v)$ and for every sort symbol s of S and for every element x of (the sorts of $A)(s)$ such that $t(p) = \langle x, s\rangle$ holds $v_1(p) = x$ and for every operation symbol o of S such that $t(p) = \langle o$, the carrier of $S\rangle$ holds $v_1(p) = (\mathrm{Den}(o,A))(\mathrm{succ}(v_1, p))$.

For simplicity, we follow the rules: S denotes a non void non empty many sorted signature, A denotes a non-empty algebra over S, V denotes a variables family of A, t denotes a term of A over V, and f denotes a many sorted function from V into the sorts of A.

The following propositions are true:

(31) Let s be a sort symbol of S and x be an element of (the sorts of $A)(s)$. Suppose $t =$ the root tree of $\langle x, s\rangle$. Then the root tree of x is an evaluation of t w.r.t. f.

(32) Let s be a sort symbol of S and v be an element of $V(s)$. Suppose $t =$ the root tree of $\langle v, s\rangle$. Then the root tree of $f(s)(v)$ is an evaluation of t w.r.t. f.

(33) Let o be an operation symbol of S, p be an argument sequence of o, A, and V, and q be a decorated tree yielding finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p$, and

(ii) for every natural number i and for every term t of A over V such that $i \in \operatorname{dom} p$ and $t = p(i)$ there exists a finite decorated tree v_1 such that $v_1 = q(i)$ and v_1 is an evaluation of t w.r.t. f.

Then there exists a finite decorated tree v_1 such that $v_1 = (\mathrm{Den}(o,A))$(the roots of q)-tree(q) and v_1 is an evaluation of $\mathrm{Sym}(o,$(the sorts of $A) \cup V)$-tree(p) **qua** term of A over V w.r.t. f.

(34) Let t be a term of A over V and e be a finite decorated tree. Suppose e is an evaluation of t w.r.t. f. Let p be a node of t and n be a node of e. If $n = p$, then $e{\restriction}n$ is an evaluation of $t{\restriction}p$ w.r.t. f.

(35) Let o be an operation symbol of S, p be an argument sequence of o, A, and V, and v_1 be a finite decorated tree. Suppose v_1 is an evaluation of $\mathrm{Sym}(o,$(the sorts of $A) \cup V)$-tree(p) **qua** term of A over V w.r.t. f. Then there exists a decorated tree yielding finite sequence q such that

(i) $\operatorname{len} q = \operatorname{len} p$,

(ii) $v_1 = (\mathrm{Den}(o,A))$(the roots of q)-tree(q), and

(iii) for every natural number i and for every term t of A over V such that $i \in \operatorname{dom} p$ and $t = p(i)$ there exists a finite decorated tree v_1 such that $v_1 = q(i)$ and v_1 is an evaluation of t w.r.t. f.

(36) There exists a finite decorated tree which is an evaluation of t w.r.t. f.

(37) Let e_1, e_2 be finite decorated trees. Suppose e_1 is an evaluation of t w.r.t. f and e_2 is an evaluation of t w.r.t. f. Then $e_1 = e_2$.

(38) Let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of t w.r.t. f. Then $v_1(\emptyset) \in$ (the sorts of A)(the sort of t).

Let S be a non void non empty many sorted signature, let A be a non-empty algebra over S, let V be a variables family of A, let t be a term of A over V, and let f be a many sorted function from V into the sorts of A. The functor $t\ {}^{@}\ f$ yielding an element of (the sorts of A)(the sort of t) is defined by:

(Def. 10) There exists a finite decorated tree v_1 such that v_1 is an evaluation of t w.r.t. f and $t\ {}^{@}\ f = v_1(\emptyset)$.

In the sequel t is a term of A over V.
We now state several propositions:

(39) For every finite decorated tree v_1 such that v_1 is an evaluation of t w.r.t. f holds $t\ {}^{@}\ f = v_1(\emptyset)$.

(40) Let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of t w.r.t. f. Let p be a node of t. Then $v_1(p) = t{\restriction}p\ {}^{@}\ f$.

(41) For every sort symbol s of S and for every element x of (the sorts of A)(s) holds $x_{A,V}\ {}^{@}\ f = x$.

(42) For every sort symbol s of S and for every element v of $V(s)$ holds $v_A\ {}^{@}\ f = f(s)(v)$.

(43) Let o be an operation symbol of S, p be an argument sequence of o, A, and V, and q be a finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p$, and

(ii) for every natural number i such that $i \in \operatorname{dom} p$ and for every term t of A over V such that $t = p(i)$ holds $q(i) = t\ {}^{@}\ f$.

Then (Sym(o,(the sorts of A) $\cup$ V)-tree(p) **qua** term of A over V) ${}^{@}\ f = (\mathrm{Den}(o,A))(q)$.

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