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Inverse Limits of Many Sorted Algebras

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Summary. This article introduces the construction of an inverse limit of many sorted algebras. A few preliminary notions such as an ordered family of many sorted algebras and a binding of family are formulated. Definitions of a set of many sorted signatures and a set of signature morphisms are also given.

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The articles [17], [11], [23], [18], [24], [8], [26], [9], [5], [22], [12], [19], [25], [10], [2], [7], [1], [3], [20], [15], [21], [6], [14], [16], [4], and [13] provide the notation and terminology for this paper.

1. Inverse Limits of Many Sorted Algebras

We follow the rules: P denotes a non empty poset, i, j, k denote elements of P, and S denotes a non void non empty many sorted signature.

Let I be a non empty set, let us consider S, let A_1 be an algebra family of I over S, let i be an element of I, and let o be an operation symbol of S. Note that $(\mathrm{OPER}(A_1))(i)(o)$ is function-like and relation-like.

Let I be a non empty set, let us consider S, let A_1 be an algebra family of I over S, and let s be a sort symbol of S. One can verify that $(\mathrm{SORTS}(A_1))(s)$ is functional.

Let us consider P, S. An algebra family of the carrier of P over S is said to be a family of algebras over S ordered by P if it satisfies the condition (Def. 1).

(Def. 1) There exists a many sorted function F indexed by the internal relation of P such that for all i, j, k if $i \geq j$ and $j \geq k$, then there exists a many sorted function f_1 from $\mathrm{it}(i)$ into $\mathrm{it}(j)$ and there exists a many sorted function f_2 from $\mathrm{it}(j)$ into $\mathrm{it}(k)$ such that $f_1 = F(j, i)$ and $f_2 = F(k, j)$ and $F(k, i) = f_2 \circ f_1$ and f_1 is a homomorphism of $\mathrm{it}(i)$ into $\mathrm{it}(j)$.

In the sequel O_1 is a family of algebras over S ordered by P.

Let us consider P, S, O_1. A many sorted function indexed by the internal relation of P is said to be a binding of O_1 if it satisfies the condition (Def. 2).

(Def. 2) Let given i, j, k. Suppose $i \geq j$ and $j \geq k$. Then there exists a many sorted function f_1 from $O_1(i)$ into $O_1(j)$ and there exists a many sorted function f_2 from $O_1(j)$ into $O_1(k)$ such that $f_1 = \mathrm{it}(j, i)$ and $f_2 = \mathrm{it}(k, j)$ and $\mathrm{it}(k, i) = f_2 \circ f_1$ and f_1 is a homomorphism of $O_1(i)$ into $O_1(j)$.

Let us consider P, S, O_1, let B be a binding of O_1, and let us consider i, j. Let us assume that $i \geq j$. The functor $\mathrm{bind}(B, i, j)$ yielding a many sorted function from $O_1(i)$ into $O_1(j)$ is defined by:

(Def. 3) $\mathrm{bind}(B, i, j) = B(j, i)$.

In the sequel B is a binding of O_1.
The following proposition is true

(1) If $i \geq j$ and $j \geq k$, then $\mathrm{bind}(B, j, k) \circ \mathrm{bind}(B, i, j) = \mathrm{bind}(B, i, k)$.

Let us consider P, S, O_1 and let I_1 be a binding of O_1. We say that I_1 is normalized if and only if:

(Def. 4) For every i holds $I_1(i, i) = \mathrm{id}_{\text{the sorts of } O_1(i)}$.

One can prove the following proposition

(2) Let given P, S, O_1, B, i, j. Suppose $i \geq j$. Let f be a many sorted function from $O_1(i)$ into $O_1(j)$. If $f = \mathrm{bind}(B, i, j)$, then f is a homomorphism of $O_1(i)$ into $O_1(j)$.

Let us consider P, S, O_1, B. The functor Normalized(B) yielding a binding of O_1 is defined by:

(Def. 5) For all i, j such that $i \geq j$ holds $(\mathrm{Normalized}(B))(j, i) = (j = i \rightarrow \mathrm{id}_{\text{the sorts of } O_1(i)}, \mathrm{bind}(B, i, j) \circ \mathrm{id}_{\text{the sorts of } O_1(i)})$.

The following proposition is true

(3) For all i, j such that $i \geq j$ and $i \neq j$ holds $B(j, i) = (\mathrm{Normalized}(B))(j, i)$.

Let us consider P, S, O_1, B. One can check that Normalized(B) is normalized.
Let us consider P, S, O_1. One can verify that there exists a binding of O_1 which is normalized.
The following proposition is true

(4) For every normalized binding N_1 of O_1 and for all i, j such that $i \geq j$ holds $(\mathrm{Normalized}(N_1))(j, i) = N_1(j, i)$.

Let us consider P, S, O_1 and let B be a binding of O_1. The functor $\underleftarrow{\lim} B$ yielding a strict subalgebra of $\prod O_1$ is defined by the condition (Def. 6).

(Def. 6) Let s be a sort symbol of S and f be an element of $(\mathrm{SORTS}(O_1))(s)$. Then $f \in$ (the sorts of $\underleftarrow{\lim} B)(s)$ if and only if for all i, j such that $i \geq j$ holds $(\mathrm{bind}(B, i, j))(s)(f(i)) = f(j)$.

Next we state the proposition

(5) Let D_1 be a discrete non empty poset, given S, O_1 be a family of algebras over S ordered by D_1, and B be a normalized binding of O_1. Then $\underleftarrow{\lim} B = \prod O_1$.

2. Sets and Morphisms of Many Sorted Signatures

In the sequel x denotes a set and A denotes a non empty set.
Let X be a set. We say that X is MSS-membered if and only if:

(Def. 7) If $x \in X$, then x is a strict non empty non void many sorted signature.

One can check that there exists a set which is non empty and MSS-membered.
The strict many sorted signature TrivialMSSign is defined as follows:

(Def. 8) TrivialMSSign is empty and void.

Let us note that TrivialMSSign is empty and void.
One can check that there exists a many sorted signature which is strict, empty, and void.
The following proposition is true

(6) Let S be a void many sorted signature. Then $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$ form morphism between S and S.

Let us consider A. The functor MSS-set(A) is defined by the condition (Def. 9).

(Def. 9) $x \in$ MSS-set(A) if and only if there exists a strict non empty non void many sorted signature S such that $x = S$ and the carrier of $S \subseteq A$ and the operation symbols of $S \subseteq A$.

Let us consider A. Observe that MSS-set(A) is non empty and MSS-membered.

Let A be a non empty MSS-membered set. We see that the element of A is a strict non empty non void many sorted signature.

Let S_1, S_2 be many sorted signatures. The functor MSS-morph(S_1, S_2) is defined as follows:

(Def. 10) $x \in$ MSS-morph(S_1, S_2) iff there exist functions f, g such that $x = \langle f, g \rangle$ and f and g form morphism between S_1 and S_2.

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