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Introduction to Modal Propositional Logic

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The articles [13], [12], [8], [16], [15], [11], [14], [17], [5], [18], [4], [9], [6], [10], [1], [7], [2], [3], and [19] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x denotes a set, n, m, k denote natural numbers, t_1 denotes a tree decorated with elements of $[:\mathbb{N}, \mathbb{N}:]$, w, s, t denote finite sequences of elements of $\mathbb{N}$, and D denotes a non empty set.

Let Z be a tree. The root of Z yielding an element of Z is defined as follows:

(Def. 1) The root of $Z = \emptyset$.

Let us consider D and let T be a tree decorated with elements of D. The root of T yielding an element of D is defined by:

(Def. 2) The root of $T = T$(the root of dom T).

The following propositions are true:

(3)[1] If $n \neq m$, then $\langle n \rangle$ and $\langle m \rangle \frown s$ are not $\subseteq$-comparable.

(4) For every s such that $s \neq \emptyset$ there exist w, n such that $s = \langle n \rangle \frown w$.

(5) If $n \neq m$, then $\langle n \rangle \not\prec \langle m \rangle \frown s$.

(6) If $n \neq m$, then $\langle n \rangle \not\preceq \langle m \rangle \frown s$.

(7) $\langle n \rangle \not\prec \langle m \rangle$.

(9)[2] The elementary tree of $1 = \{\emptyset, \langle 0 \rangle\}$.

(10) The elementary tree of $2 = \{\emptyset, \langle 0 \rangle, \langle 1 \rangle\}$.

(11) For every tree Z and for all n, m such that $n \leq m$ and $\langle m \rangle \in Z$ holds $\langle n \rangle \in Z$.

(12) If $w \frown t \prec w \frown s$, then $t \prec s$.

(13) $t_1 \in \mathbb{N}^* \dot{\rightarrow} [:\mathbb{N}, \mathbb{N}:]$.

(15)[3] For all trees Z, Z_1, Z_2 and for every element z of Z such that Z with-replacement$(z, Z_1) = Z$ with-replacement(z, Z_2) holds $Z_1 = Z_2$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (8) has been removed.
[3] The proposition (14) has been removed.

(16) For all trees Z, Z_1, Z_2 decorated with elements of D and for every element z of $\operatorname{dom} Z$ such that Z with-replacement$(z, Z_1) = Z$ with-replacement(z, Z_2) holds $Z_1 = Z_2$.

(17) Let Z_1, Z_2 be trees and p be a finite sequence of elements of $\mathbb{N}$. Suppose $p \in Z_1$. Let v be an element of Z_1 with-replacement(p, Z_2) and w be an element of Z_1. If $v = w$ and $w \prec p$, then $\operatorname{succ} v = \operatorname{succ} w$.

(18) Let Z_1, Z_2 be trees and p be a finite sequence of elements of $\mathbb{N}$. Suppose $p \in Z_1$. Let v be an element of Z_1 with-replacement(p, Z_2) and w be an element of Z_1. If $v = w$ and p and w are not $\subseteq$-comparable, then $\operatorname{succ} v = \operatorname{succ} w$.

(19) Let Z_1, Z_2 be trees and p be a finite sequence of elements of $\mathbb{N}$. Suppose $p \in Z_1$. Let v be an element of Z_1 with-replacement(p, Z_2) and w be an element of Z_2. If $v = p \frown w$, then $\operatorname{succ} v \approx \operatorname{succ} w$.

(20) Let Z_1 be a tree and p be a finite sequence of elements of $\mathbb{N}$. Suppose $p \in Z_1$. Let v be an element of Z_1 and w be an element of $Z_1 \restriction p$. If $v = p \frown w$, then $\operatorname{succ} v \approx \operatorname{succ} w$.

(22)[4] For every finite tree Z such that the branch degree of the root of $Z = 0$ holds $\operatorname{card} Z = 1$ and $Z = \{\emptyset\}$.

(23) For every finite tree Z such that the branch degree of the root of $Z = 1$ holds succ(the root of $Z) = \{\langle 0 \rangle\}$.

(24) For every finite tree Z such that the branch degree of the root of $Z = 2$ holds succ(the root of $Z) = \{\langle 0 \rangle, \langle 1 \rangle\}$.

In the sequel s', w' are elements of $\mathbb{N}^*$.

Next we state several propositions:

(25) Let Z be a tree and o be an element of Z. Suppose $o \neq$ the root of Z. Then $Z \restriction o \approx \{o \frown s' : o \frown s' \in Z\}$ and the root of $Z \notin \{o \frown w' : o \frown w' \in Z\}$.

(26) For every finite tree Z and for every element o of Z such that $o \neq$ the root of Z holds $\operatorname{card}(Z \restriction o) < \operatorname{card} Z$.

(27) Let Z be a finite tree and z be an element of Z. If succ(the root of $Z) = \{z\}$, then $Z =$ (the elementary tree of 1) with-replacement$(\langle 0 \rangle, Z \restriction z)$.

(28) Let Z be a finite tree decorated with elements of D and z be an element of $\operatorname{dom} Z$. Suppose succ(the root of $\operatorname{dom} Z) = \{z\}$ and $\operatorname{dom} Z$ is finite. Then $Z =$ ((the elementary tree of 1) $\longmapsto$ (the root of Z)) with-replacement$(\langle 0 \rangle, Z \restriction z)$.

(29) Let Z be a tree and x_1, x_2 be elements of Z. Suppose Z is finite and $x_1 = \langle 0 \rangle$ and $x_2 = \langle 1 \rangle$ and succ(the root of $Z) = \{x_1, x_2\}$. Then $Z =$ (the elementary tree of 2) with-replacement$(\langle 0 \rangle, Z \restriction x_1)$ with-replacement$(\langle 1 \rangle, Z \restriction x_2)$.

(30) Let Z be a tree decorated with elements of D and x_1, x_2 be elements of $\operatorname{dom} Z$. Suppose $\operatorname{dom} Z$ is finite and $x_1 = \langle 0 \rangle$ and $x_2 = \langle 1 \rangle$ and succ(the root of $\operatorname{dom} Z) = \{x_1, x_2\}$. Then $Z =$ ((the elementary tree of 2) $\longmapsto$ (the root of Z)) with-replacement$(\langle 0 \rangle, Z \restriction x_1)$ with-replacement$(\langle 1 \rangle, Z \restriction x_2)$.

The set $\mathcal{V}$ is defined as follows:

(Def. 3) $\mathcal{V} = [: \{3\}, \mathbb{N} :]$.

Let us observe that $\mathcal{V}$ is non empty.

A variable is an element of $\mathcal{V}$.

The set $\mathcal{C}$ is defined as follows:

[4] The proposition (21) has been removed.

(Def. 4) $\mathcal{C} = [:\{0,1,2\}, \mathbb{N}:]$.

Let us mention that $\mathcal{C}$ is non empty.
A connective is an element of $\mathcal{C}$.
One can prove the following proposition

(31) $\mathcal{C}$ misses $\mathcal{V}$.

In the sequel p, q denote variables.

Let T be a finite tree and let v be an element of T. Then the branch degree of v is a natural number.

Let D be a non empty set. A non empty set is called a non empty set of trees decorated with elements of D if:

(Def. 5) For every x such that $x \in$ it holds x is a tree decorated with elements of D.

Let D_0 be a non empty set and let D be a non empty set of trees decorated with elements of D_0. We see that the element of D is a tree decorated with elements of D_0.

The non empty set WFF of trees decorated with elements of $[:\mathbb{N}, \mathbb{N}:]$ is defined by the conditions (Def. 6).

(Def. 6)(i) For every tree x decorated with elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $x \in$ WFF holds x is finite, and

(ii) for every finite tree x decorated with elements of $[:\mathbb{N}, \mathbb{N}:]$ holds $x \in$ WFF iff for every element v of dom$x$ holds the branch degree of $v \leq 2$ and if the branch degree of $v = 0$, then $x(v) = \langle 0, 0 \rangle$ or there exists k such that $x(v) = \langle 3, k \rangle$ and if the branch degree of $v = 1$, then $x(v) = \langle 1, 0 \rangle$ or $x(v) = \langle 1, 1 \rangle$ and if the branch degree of $v = 2$, then $x(v) = \langle 2, 0 \rangle$.

A MP-formula is an element of WFF.
Let us observe that every MP-formula is finite.
In the sequel A, A_1, B, B_1, C denote MP-formulae.
Let us consider A and let a be an element of dom$A$. Then $A \upharpoonright a$ is a MP-formula.
Let a be an element of $\mathcal{C}$. The functor Arity(a) yielding a natural number is defined by:

(Def. 7) Arity$(a) = a_{\mathbf{1}}$.

Let D be a non empty set, let T, T_1 be trees decorated with elements of D, and let p be a finite sequence of elements of $\mathbb{N}$. Let us assume that $p \in \mathrm{dom}\, T$. The functor $T(p \leftarrow T_1)$ yields a tree decorated with elements of D and is defined as follows:

(Def. 8) $T(p \leftarrow T_1) = T$ with-replacement(p, T_1).

One can prove the following propositions:

(32) ((The elementary tree of 1) $\longmapsto \langle 1, 0 \rangle$) with-replacement$(\langle 0 \rangle, A)$ is a MP-formula.

(33) ((The elementary tree of 1) $\longmapsto \langle 1, 1 \rangle$) with-replacement$(\langle 0 \rangle, A)$ is a MP-formula.

(34) ((The elementary tree of 2) $\longmapsto \langle 2, 0 \rangle$) with-replacement$(\langle 0 \rangle, A)$ with-replacement$(\langle 1 \rangle, B)$ is a MP-formula.

Let us consider A. The functor $\neg A$ yielding a MP-formula is defined by:

(Def. 9) $\neg A$ = ((the elementary tree of 1) $\longmapsto \langle 1, 0 \rangle$) with-replacement$(\langle 0 \rangle, A)$.

The functor $\Box A$ yielding a MP-formula is defined by:

(Def. 10) $\Box A$ = ((the elementary tree of 1) $\longmapsto \langle 1, 1 \rangle$) with-replacement$(\langle 0 \rangle, A)$.

Let us consider B. The functor $A \wedge B$ yields a MP-formula and is defined by:

(Def. 11) $A \wedge B$ = ((the elementary tree of 2) $\longmapsto \langle 2, 0 \rangle$) with-replacement$(\langle 0 \rangle, A)$ with-replacement$(\langle 1 \rangle, B)$.

Let us consider A. The functor $\Diamond A$ yields a MP-formula and is defined as follows:

(Def. 12) $\Diamond A = \neg\Box\neg A$.

Let us consider B. The functor $A \vee B$ yielding a MP-formula is defined by:

(Def. 13) $A \vee B = \neg(\neg A \wedge \neg B)$.

The functor $A \Rightarrow B$ yielding a MP-formula is defined by:

(Def. 14) $A \Rightarrow B = \neg(A \wedge \neg B)$.

The following two propositions are true:

(35) (The elementary tree of 0) $\longmapsto \langle 3, n \rangle$ is a MP-formula.

(36) (The elementary tree of 0) $\longmapsto \langle 0, 0 \rangle$ is a MP-formula.

Let us consider p. The functor ${}^{@}p$ yields a MP-formula and is defined as follows:

(Def. 15) ${}^{@}p =$ (the elementary tree of 0) $\longmapsto p$.

One can prove the following propositions:

(37) If ${}^{@}p = {}^{@}q$, then $p = q$.

(38) If $\neg A = \neg B$, then $A = B$.

(39) If $\Box A = \Box B$, then $A = B$.

(40) If $A \wedge B = A_1 \wedge B_1$, then $A = A_1$ and $B = B_1$.

The MP-formula VERUM is defined as follows:

(Def. 16) VERUM $=$ (the elementary tree of 0) $\longmapsto \langle 0, 0 \rangle$.

The following propositions are true:

(42)[5] If $\mathrm{card}\,\mathrm{dom} A = 1$, then $A =$ VERUM or there exists p such that $A = {}^{@}p$.

(43) If $\mathrm{card}\,\mathrm{dom} A \geq 2$, then there exists B such that $A = \neg B$ or $A = \Box B$ or there exist B, C such that $A = B \wedge C$.

(44) $\mathrm{card}\,\mathrm{dom} A < \mathrm{card}\,\mathrm{dom} \neg A$.

(45) $\mathrm{card}\,\mathrm{dom} A < \mathrm{card}\,\mathrm{dom} \Box A$.

(46) $\mathrm{card}\,\mathrm{dom} A < \mathrm{card}\,\mathrm{dom}(A \wedge B)$ and $\mathrm{card}\,\mathrm{dom} B < \mathrm{card}\,\mathrm{dom}(A \wedge B)$.

Let I_1 be a MP-formula. We say that I_1 is atomic if and only if:

(Def. 17) There exists p such that $I_1 = {}^{@}p$.

We say that I_1 is negative if and only if:

(Def. 18) There exists A such that $I_1 = \neg A$.

We say that I_1 is necessitive if and only if:

(Def. 19) There exists A such that $I_1 = \Box A$.

We say that I_1 is conjunctive if and only if:

(Def. 20) There exist A, B such that $I_1 = A \wedge B$.

[5] The proposition (41) has been removed.

One can check the following observations:

* there exists a MP-formula which is atomic,
* there exists a MP-formula which is negative,
* there exists a MP-formula which is necessitive, and
* there exists a MP-formula which is conjunctive.

The scheme *MP Ind* concerns a unary predicate $\mathcal{P}$, and states that:
For every element A of WFF holds $\mathcal{P}[A]$
provided the following conditions are satisfied:
- $\mathcal{P}[\text{VERUM}]$,
- For every variable p holds $\mathcal{P}[{}^{@}p]$,
- For every element A of WFF such that $\mathcal{P}[A]$ holds $\mathcal{P}[\neg A]$,
- For every element A of WFF such that $\mathcal{P}[A]$ holds $\mathcal{P}[\Box A]$, and
- For all elements A, B of WFF such that $\mathcal{P}[A]$ and $\mathcal{P}[B]$ holds $\mathcal{P}[A \wedge B]$.

Next we state several propositions:

(47) Let A be an element of WFF. Then
(i) $A = \text{VERUM}$, or
(ii) A is an atomic MP-formula, a negative MP-formula, a necessitive MP-formula, and a conjunctive MP-formula.

(48) $A = \text{VERUM}$ or there exists p such that $A = {}^{@}p$ or there exists B such that $A = \neg B$ or there exists B such that $A = \Box B$ or there exist B, C such that $A = B \wedge C$.

(49) ${}^{@}p \neq \neg A$ and ${}^{@}p \neq \Box A$ and ${}^{@}p \neq A \wedge B$.

(50) $\neg A \neq \Box B$ and $\neg A \neq B \wedge C$.

(51) $\Box A \neq B \wedge C$.

(52) $\text{VERUM} \neq {}^{@}p$ and $\text{VERUM} \neq \neg A$ and $\text{VERUM} \neq \Box A$ and $\text{VERUM} \neq A \wedge B$.

The scheme *MP Func Ex* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, two unary functors $\mathcal{G}$ and $\mathcal{H}$ yielding elements of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and states that:
There exists a function f from WFF into $\mathcal{A}$ such that
(i) $f(\text{VERUM}) = \mathcal{B}$,
(ii) for every variable p holds $f({}^{@}p) = \mathcal{F}(p)$,
(iii) for every element A of WFF holds $f(\neg A) = \mathcal{G}(f(A))$,
(iv) for every element A of WFF holds $f(\Box A) = \mathcal{H}(f(A))$, and
(v) for all elements A, B of WFF holds $f(A \wedge B) = \mathcal{I}(f(A), f(B))$
for all values of the parameters.

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