JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

Free Modules

Michał Muzalewski
Warsaw University
Białystok

Summary. We define free modules and prove that every left module over Skew-Field is free.

MML Identifier: MOD_3.

WWW: http://mizar.org/JFM/Vol3/mod_3.html

The articles [10], [5], [17], [6], [2], [18], [3], [4], [11], [12], [1], [13], [7], [8], [16], [15], [14], and [9] provide the notation and terminology for this paper.

One can prove the following proposition

(2)[1] Let R be a non degenerated add-associative right zeroed right complementable non empty double loop structure. Then $0_R \neq -\mathbf{1}_R$.

For simplicity, we adopt the following rules: x denotes a set, R denotes a ring, V denotes a left module over R, L denotes a linear combination of V, a denotes a scalar of R, F denotes a finite sequence of elements of the carrier of V, and C denotes a finite subset of V.

One can prove the following two propositions:

(6)[2] If the support of $L \subseteq C$, then there exists F such that F is one-to-one and $\operatorname{rng} F = C$ and $\sum L = \sum(L\,F)$.

(7) $\sum(a \cdot L) = a \cdot \sum L$.

In the sequel A, B are subsets of V and l is a linear combination of A.

Let us consider R, V, A. The functor $\operatorname{Lin}(A)$ yielding a strict subspace of V is defined by:

(Def. 1) The carrier of $\operatorname{Lin}(A) = \{\sum l\}$.

One can prove the following propositions:

(11)[3] $x \in \operatorname{Lin}(A)$ iff there exists l such that $x = \sum l$.

(12) If $x \in A$, then $x \in \operatorname{Lin}(A)$.

(13) $\operatorname{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(14) If $\operatorname{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.

(15) For every strict subspace W of V such that $0_R \neq \mathbf{1}_R$ and $A =$ the carrier of W holds $\operatorname{Lin}(A) = W$.

[1] The proposition (1) has been removed.
[2] The propositions (3)–(5) have been removed.
[3] The propositions (8)–(10) have been removed.

© Association of Mizar Users

(16) Let V be a strict left module over R and A be a subset of V. If $0_R \neq \mathbf{1}_R$ and $A =$ the carrier of V, then $\mathrm{Lin}(A) = V$.

(17) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(18) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(19) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(20) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

Let us consider R, V and let I_1 be a subset of V. We say that I_1 is base if and only if:

(Def. 2) I_1 is linearly independent and $\mathrm{Lin}(I_1) =$ the vector space structure of V.

Let us consider R and let I_1 be a left module over R. We say that I_1 is free if and only if:

(Def. 3) There exists a subset of I_1 which is base.

The following proposition is true

(21) $\mathbf{0}_V$ is free.

Let us consider R. Observe that there exists a left module over R which is strict and free.

For simplicity, we adopt the following rules: R is a skew field, a, b are scalars of R, V is a left module over R, and v, v_1, v_2 are vectors of V.

The following propositions are true:

(23)[4] $\{v\}$ is linearly independent iff $v \neq 0_V$.

(24) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff $v_2 \neq 0_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(25) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0_R$ and $b = 0_R$.

(26) Let V be a left module over R and A be a subset of V. Suppose A is linearly independent. Then there exists a subset B of V such that $A \subseteq B$ and B is base.

(27) Let V be a left module over R and A be a subset of V. If $\mathrm{Lin}(A) = V$, then there exists a subset B of V such that $B \subseteq A$ and B is base.

(28) Every left module over R is free.

Let us consider R and let V be a left module over R. A subset of V is called a basis of V if:

(Def. 5)[5] It is base.

One can prove the following two propositions:

(29) Let V be a left module over R and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(30) For every left module V over R and for every subset A of V such that $\mathrm{Lin}(A) = V$ there exists a basis I of V such that $I \subseteq A$.

[4] The proposition (22) has been removed.

[5] The definition (Def. 4) has been removed.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[6] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[8] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_2.html.

[9] Michał Muzalewski and Wojciech Skaba. Linear independence in left module over domain. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/lmod_5.html.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[12] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[13] Wojciech A. Trybulec. Linear combinations in real linear space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rlvect_2.html.

[14] Wojciech A. Trybulec. Linear combinations in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_6.html.

[15] Wojciech A. Trybulec. Operations on subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_5.html.

[16] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_4.html.

[17] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[18] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received October 18, 1991

Published January 2, 2004