JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Reper Algebras

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Summary. We shall describe n-dimensional spaces with the reper operation [7, pages 72–79]. An inspiration to such approach comes from the monograph [9] and so-called Leibniz program. Let us recall that the Leibniz program is a program of algebraization of geometry using purely geometric notions. Leibniz formulated his program in opposition to algebraization method developed by Descartes. The Euclidean geometry in Szmielew's approach [9] is a theory of structures $\langle S;\, \|, \oplus, O\rangle$, where $\langle S;\, \|, \oplus, O\rangle$ is Desarguesian midpoint plane and $O \subseteq S \times S \times S$ is the relation of equi-orthogonal basis. Points o, p, q are in relation O if they form an isosceles triangle with the right angle in vertex a. If we fix vertices a, p, then there exist exactly two points q, q' such that $O(apq)$, $O(apq')$. Moreover $q \oplus q' = a$. In accordance with the Leibniz program we replace the relation of equi-orthogonal basis by a binary operation $*: S \times S \to S$, called the reper operation. A standard model for the Euclidean geometry in the above sense is the oriented plane over the field of real numbers with the reper operations $*$ defined by the condition: $a * b = q$ iff the point q is the result of rotating of p about right angle around the center a.

MML Identifier: `MIDSP_3`.

WWW: `http://mizar.org/JFM/Vol4/midsp_3.html`

The articles [10], [12], [3], [4], [2], [5], [1], [11], [6], and [8] provide the notation and terminology for this paper.

1. Substitutions in Tuples

For simplicity, we adopt the following rules: n, i, j, k, l are natural numbers, D is a non empty set, c, d are elements of D, and p, q, r are finite sequences of elements of D.

The following propositions are true:

(1) If $\operatorname{len} p = j+1+k$, then there exist q, r, c such that $\operatorname{len} q = j$ and $\operatorname{len} r = k$ and $p = q ^\frown \langle c\rangle ^\frown r$.

(2) If $i \in \operatorname{Seg} n$, then there exist j, k such that $n = j+1+k$ and $i = j+1$.

(3) Suppose $p = q ^\frown \langle c\rangle ^\frown r$ and $i = \operatorname{len} q + 1$. Then for every l such that $1 \leq l$ and $l \leq \operatorname{len} q$ holds $p(l) = q(l)$ and $p(i) = c$ and for every l such that $i+1 \leq l$ and $l \leq \operatorname{len} p$ holds $p(l) = r(l-i)$.

(4) $l \leq j$ or $l = j+1$ or $j+2 \leq l$.

(5) If $l \in \operatorname{Seg} n \setminus \{i\}$ and $i = j+1$, then $1 \leq l$ and $l \leq j$ or $i+1 \leq l$ and $l \leq n$.

Let us consider n, i, D, d and let p be an element of D^{n+1}. Let us assume that $i \in \operatorname{Seg}(n+1)$. The functor $p(i/d)$ yielding an element of D^{n+1} is defined by:

(Def. 1) $p(i/d)(i) = d$ and for every l such that $l \in \operatorname{dom} p \setminus \{i\}$ holds $p(i/d)(l) = p(l)$.

2. Reper Algebra Structure and its Properties

Let us consider n. We consider structures of reper algebra over n as extensions of midpoint algebra structure as systems

$\langle$ a carrier, a midpoint operation, a reper $\rangle$,

where the carrier is a set, the midpoint operation is a binary operation on the carrier, and the reper is a function from the carrier^n into the carrier.

Let us consider n, let A be a non empty set, let m be a binary operation on A, and let r be a function from A^n into A. Note that $\langle A, m, r \rangle$ is non empty.

Let us consider n. Observe that there exists a structure of reper algebra over n which is non empty.

Let us consider n. Note that there exists a non empty structure of reper algebra over $n+2$ which is midpoint algebra-like.

We follow the rules: R_1 is a midpoint algebra-like non empty structure of reper algebra over $n+2$ and a, b, d, p_1, p'_1 are points of R_1.

Let us consider i, D. A i-tuple of D is an element of D^i.

Let us consider n, R_1, i. A i-tuple of R_1 is a i-tuple of the carrier of R_1.

In the sequel p, q are $n+1$-tuples of R_1.

Let us consider n, R_1, a. Then $\langle a \rangle$ is a 1-tuple of R_1.

Let us consider n, R_1, i, j, let p be a i-tuple of R_1, and let q be a j-tuple of R_1. Then $p \frown q$ is a $i+j$-tuple of R_1.

Next we state the proposition

(6) $\langle a \rangle \frown p$ is a $n+2$-tuple of R_1.

Let us consider n, R_1, a, p. The functor $*(a, p)$ yields a point of R_1 and is defined as follows:

(Def. 2) $*(a, p) =$ (the reper of R_1)$(\langle a \rangle \frown p)$.

Let us consider n, i, R_1, d, p. The functor $p_{\restriction i \dashrightarrow d}$ yielding a $n+1$-tuple of R_1 is defined by the condition (Def. 3).

(Def. 3) Let given D, p' be an element of D^{n+1}, and d' be an element of D. If $D =$ the carrier of R_1 and $p' = p$ and $d' = d$, then $p_{\restriction i \dashrightarrow d} = p'(i/d')$.

One can prove the following proposition

(7) If $i \in \mathrm{Seg}(n+1)$, then $p_{\restriction i \dashrightarrow d}(i) = d$ and for every l such that $l \in \mathrm{dom}\, p \setminus \{i\}$ holds $p_{\restriction i \dashrightarrow d}(l) = p(l)$.

Let us consider n. A natural number is called a natural number of n if:

(Def. 4) $1 \leq$ it and it $\leq n+1$.

In the sequel m denotes a natural number of n.

The following four propositions are true:

(8) i is a natural number of n iff $i \in \mathrm{Seg}(n+1)$.

(10)[1] If $i \leq n$, then $i+1$ is a natural number of n.

(11) If for every m holds $p(m) = q(m)$, then $p = q$.

(12)(i) For every natural number l of n such that $l = i$ holds $p_{\restriction i \dashrightarrow d}(l) = d$, and

(ii) for all natural numbers l, i of n such that $l \neq i$ holds $p_{\restriction i \dashrightarrow d}(l) = p(l)$.

Let us consider n, D, let p be an element of D^{n+1}, and let us consider m. Then $p(m)$ is an element of D.

Let us consider n, R_1. We say that R_1 is invariance if and only if:

[1] The proposition (9) has been removed.

(Def. 5) For all a, b, p, q such that for every m holds $a\,^{@}\,q(m) = b\,^{@}\,p(m)$ holds $a\,^{@} * (b,q) = b\,^{@} * (a,p)$.

We introduce R_1 is invariance as a synonym of R_1 is invariance.

Let us consider n, i, R_1. We say that R_1 has property of zero in i if and only if:

(Def. 6) For all a, p holds $*(a, p_{\upharpoonright i \dot{\to} a}) = a$.

Let us consider n, i, R_1. We say that R_1 is semi additive in i if and only if:

(Def. 7) For all a, p_1, p such that $p(i) = p_1$ holds $*(a, p_{\upharpoonright i \dot{\to} a\,^{@}\,p_1}) = a\,^{@} * (a,p)$.

The following proposition is true

(13) If R_1 is semi additive in m, then for all a, d, p, q such that $q = p_{\upharpoonright m \dot{\to} d}$ holds $*(a, p_{\upharpoonright m \dot{\to} a\,^{@}\,d}) = a\,^{@} * (a,q)$.

Let us consider n, i, R_1. We say that R_1 is additive in i if and only if:

(Def. 8) For all a, p_1, p'_1, p such that $p(i) = p_1$ holds $*(a, p_{\upharpoonright i \dot{\to} p_1\,^{@}\,p'_1}) = *(a,p)\,^{@} * (a, p_{\upharpoonright i \dot{\to} p'_1})$.

Let us consider n, i, R_1. We say that R_1 is alternative in i if and only if:

(Def. 9) For all a, p, p_1 such that $p(i) = p_1$ holds $*(a, p_{\upharpoonright i+1 \dot{\to} p_1}) = a$.

In the sequel W is an atlas of R_1 and v is a vector of W.

Let us consider n, R_1, W, i. A i-tuple of W is a i-tuple of the carrier of the algebra of W.

In the sequel x, y denote $n+1$-tuples of W.

Let us consider n, R_1, W, x, i, v. The functor $x_{\upharpoonright i \dot{\to} v}$ yields a $n+1$-tuple of W and is defined by the condition (Def. 10).

(Def. 10) Let given D, x' be an element of D^{n+1}, and v' be an element of D. Suppose $D =$ the carrier of the algebra of W and $x' = x$ and $v' = v$. Then $x_{\upharpoonright i \dot{\to} v} = x'(i/v')$.

Next we state three propositions:

(14) If $i \in \mathrm{Seg}(n+1)$, then $x_{\upharpoonright i \dot{\to} v}(i) = v$ and for every l such that $l \in \mathrm{dom}\, x \setminus \{i\}$ holds $x_{\upharpoonright i \dot{\to} v}(l) = x(l)$.

(15)(i) For every natural number l of n such that $l = i$ holds $x_{\upharpoonright i \dot{\to} v}(l) = v$, and

(ii) for all natural numbers l, i of n such that $l \neq i$ holds $x_{\upharpoonright i \dot{\to} v}(l) = x(l)$.

(16) If for every m holds $x(m) = y(m)$, then $x = y$.

The scheme *SeqLambdaD'* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

There exists a finite sequence z of elements of $\mathcal{B}$ such that $\mathrm{len}\, z = \mathcal{A}+1$ and for every natural number j of $\mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

Let us consider n, R_1, W, a, x. The functor $(a,x).W$ yields a $n+1$-tuple of R_1 and is defined as follows:

(Def. 11) $((a,x).W)(m) = (a, x(m)).W$.

Let us consider n, R_1, W, a, p. The functor $W(a, p)$ yielding a $n+1$-tuple of W is defined by:

(Def. 12) $W(a, p)(m) = W(a, p(m))$.

Next we state three propositions:

(17) $W(a, p) = x$ iff $(a,x).W = p$.

(18) $W(a, (a,x).W) = x$.

(19) $(a,W(a,p)).W = p$.

Let us consider n, R_1, W, a, x. The functor $\Phi(a,x)$ yields a vector of W and is defined by:

(Def. 13) $\Phi(a,x) = W(a, *(a,(a,x).W))$.

We now state four propositions:

(20) If $W(a,p) = x$ and $W(a,b) = v$, then $*(a,p) = b$ iff $\Phi(a,x) = v$.

(21) R_1 is invariance iff for all a, b, x holds $\Phi(a,x) = \Phi(b,x)$.

(22) $1 \in \mathrm{Seg}(n+1)$.

(24)[2] 1 is a natural number of n.

3. Reper Algebra and its Atlas

Let us consider n. A midpoint algebra-like non empty structure of reper algebra over $n+2$ is said to be a reper algebra of n if:

(Def. 14) It is invariance.

For simplicity, we follow the rules: R_1 is a reper algebra of n, a, b are points of R_1, p is a $n+1$-tuple of R_1, W is an atlas of R_1, v is a vector of W, and x is a $n+1$-tuple of W.
One can prove the following proposition

(25) $\Phi(a,x) = \Phi(b,x)$.

Let us consider n, R_1, W, x. The functor $\Phi(x)$ yielding a vector of W is defined as follows:

(Def. 15) For every a holds $\Phi(x) = \Phi(a,x)$.

Next we state a number of propositions:

(26) If $W(a,p) = x$ and $W(a,b) = v$ and $\Phi(x) = v$, then $*(a,p) = b$.

(27) If $(a,x).W = p$ and $(a,v).W = b$ and $*(a,p) = b$, then $\Phi(x) = v$.

(28) If $W(a,p) = x$ and $W(a,b) = v$, then $W(a, p_{\restriction m \dot{\to} b}) = x_{\restriction m \dot{\to} v}$.

(29) If $(a,x).W = p$ and $(a,v).W = b$, then $(a, x_{\restriction m \dot{\to} v}).W = p_{\restriction m \dot{\to} b}$.

(30) R_1 has property of zero in m iff for every x holds $\Phi((x_{\restriction m \dot{\to} 0_W})) = 0_W$.

(31) R_1 is semi additive in m iff for every x holds $\Phi((x_{\restriction m \dot{\to} 2 x(m)})) = 2\,\Phi(x)$.

(32) If R_1 is additive in m and has property of zero in m, then R_1 is semi additive in m.

(33) If R_1 has property of zero in m, then R_1 is additive in m iff for all x, v holds $\Phi((x_{\restriction m \dot{\to} x(m)+v})) = \Phi(x) + \Phi((x_{\restriction m \dot{\to} v}))$.

(34) If $W(a,p) = x$ and $m \leq n$, then $W(a, p_{\restriction m+1 \dot{\to} p(m)}) = x_{\restriction m+1 \dot{\to} x(m)}$.

(35) If $(a,x).W = p$ and $m \leq n$, then $(a, x_{\restriction m+1 \dot{\to} x(m)}).W = p_{\restriction m+1 \dot{\to} p(m)}$.

(36) If $m \leq n$, then R_1 is alternative in m iff for every x holds $\Phi((x_{\restriction m+1 \dot{\to} x(m)})) = 0_W$.

[2] The proposition (23) has been removed.

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Received May 28, 1992

Published January 2, 2004